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THEORY OF OPTICAL IMAGE IMPROVEMENT

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This report outlines a general theory of feedback systems designed to compensate the image-degrading effects of atmospheric seeing in real time. "Real time" means that the optical surfaces must move rapidly enough to keep the image in sharp focus at each instant. The report demonstrates that a system producing diffraction-limited resolution in a large ground-based telescope is possible, in principle. The field of sharply focused view is very narrow (of the order of 10

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seconds of arc), and the system can work only with moderately bright objects (about magnitude 10 or brighter). Section I presents a general discussion and summary of results. Section II continues with a rigorous solution of the problem of the design of an optimum feedback system for stabilizing the phases in a simple two-beam optical interferometer perturbed by a fluctuating atmosphere. The same method of optimization is applied to the design of complete image-improvement systems in the remainder of the report.

In Sections III through VI a system for real-time correction of atmospheric seeing in an optical telescope is described in detail. The system collects all the information available from a monitor of the telescope image, and uses this information to control the optical surfaces so as to compensate for atmospheric distortion of the wavefront. Among a large class of possible programs for feeding back information from the image to the mirror surfaces, the optimum program is defined and computed. The optimized system is predicted to give diffraction-limited images with a mean-square phase error (measured in radians) $D = f[p^4\eta^2\phi/\nu d^2]^{-1/5}$, where f is a numerical coefficient approximately equal to 1/2, p is the patch size and ν is the bandwidth of atmospheric disturbances, ϕ is the total flux of detected photons, η is the fraction of incident light belonging to components of sharply defined structure, and d is the diameter of the telescope aperture. Diffraction-limited images of acceptable quality should be possible when the ratio in square brackets in this formula is greater than unity.

Sections VII through X cover the same ground as Sections III through VI. No new physical principle is introduced, but the theory of single-image optical improvement systems is recast into a form that is mathematically more precise and more general. Exact expressions are found for the optimum feedback program in a single-image system of arbitrary geometry (Section VII). It is rigorously proved that the mirror motions in an optimized system can be represented by a finite set of normal modes, the number of modes depending on the behavior of the atmosphere and the optics but having nothing to do with the mechanical construction of the mirror (Section IX). In Appendix F, containing the hard mathematics, it is shown that the integral equations determining the optimum feedback program are formally identical with the Gelfand-Levitan equations of inverse scattering theory. The analogy between image improvement and the Gelfand-Levitan theory is more than formal, since the reconstruction of atmospheric disturbances from their effects on the image is in essence an inverse scattering problem.

CONTENTS

APPENDICES (Continued)

I INTRODUCTION AND SUMMARY

In recent years there have been several successful efforts to extract from ground-based telescopic images some of the information that atmospheric "seeing" (degradation of the image by atmospheric inhomogeneities) destroys in normal astronomical photography. The best-known technique of information-recovery is the "speckle interferometry" of Labeyrie (1970). A survey of such techniques was recently published by Code (1973). All the techniques that have so far been developed have modest aims. They can at best recover only a small fraction of the lost information. On the other hand, Babcock (1953) proposed twenty years ago a system aimed at the total restoration of images. Babcock proposed to build a telescope with an actively mobile optical system, capable of compensating completely the effects of atmospheric seeing in real time. I do not know whether Babcock's ideas received any serious attention from telescope designers in the 1950's. It would have been very difficult to implement Babcock's scheme twenty years ago, because we lacked the technology of on-line computers. Since on-line computers and efficient image-monitoring devices have now become available, the time is ripe for a serious study of the feasibility of Babcock's proposal. The purpose of the present series of papers is to provide a theoretical basis for such a study. My collaborator, Dr. Richard Muller (1974), is simultaneously carrying out computer simulations and designing optical equipment that will constitute the experimental part of the study.

During the last few years several groups, apparently unaware of Babcock's paper, have independently made proposals of active optical systems that are in essence modernized versions of Babcock's system. I am grateful to Dr. D. L. Fried of Optical Science Consultants, and to Dr. R. S. Leonard of Stanford Research Institute, for information about two such proposals. I will not discuss these proposals in detail since they involve company proprietary information. I only wish to acknowledge that they provided the initial stimulus that led me to develop the theory presented in these papers.

The image-improvement concept of Babcock involves four essential components: (1) A mirror or optical correcting-plate, composed of elements that can be moved rapidly and independently in a direction normal to the surface, (2) an image-detector or television camera system that records instantaneously the positions and times of arrival of photons as they reach the image or images, (3) a data-processing system with an on-line computer that accepts data from the image-detectors and calculates the desired response of the various elements of the mirror, and (4) a servosystem that moves the mirror elements as directed by the computer. Of these four components, three (1, 2, and 4) are hardware items whose existence I shall take for granted. Their implementation is a problem for the experimenter rather than the theorist. Since the amplitude of motion required of the mirror elements is only a few optical wavelengths (of the order of 10^{-4} cm), and the atmospheric fluctuations to which they are required to respond have frequencies in the range from 1 to 1000 Hz, the required velocities are only of the order of millimeters per second and should be easily attainable by means of piezoelectric crystals or other conventional transducers. In these papers I shall be primarily concerned with Item 3, the software item. The essential question that has to be answered is the following: What kind of program should be given to the computer, to convert information arriving at the image into

instructions telling the mirror what to do to improve the quality of the image? It is not self-evident that any program exists that will enable the image to lift itself up by its bootstraps in this fashion. We shall outline the construction of such a program, and show that images of substantially improved resolution can in principle by obtained by using it. How great the improvement is, and how wide the class of astronomical object that is accessible to this technique, can only be determined by operational trials.

The compensation of atmospheric seeing is possible only under two severe limitations. First, the highly-resolved optical field must be very narrow, at best like a tiny fovea in the visual field of the human eye. The reason for the narrowness of the field is that a single configuration of the mirror surface is supposed to compensate for the distortion of the wavefronts of all photons arriving at the mirror. This is possible only if all photons arriving at a given point on the mirror pass through the same atmospheric inhomogeneities. Therefore the angular size of the correctly-compensated visual field is equal to the angular size of the disturbing patches in the atmosphere as seen from the telescope. Estimates of the size of the patches range from seconds to minutes of arc; some of the relevant observational data are summarized by Bufton and Genatt (1971). The wide scatter of the data indicates that patch sizes are highly variable from place to place and from night to night. One of the byproducts of an operational image-improvement system will be accurate information about the scale and spectrum of atmospheric inhomogeneities.

The second basic limitation on the compensation of seeing is the problem of photon statistics. There must be enough photons arriving at each resolution element of the image during each atmospheric fluctuation period, to provide reliable information about the momentary distortion

of the wavefronts. The image-improvement system is therefore in principle limited not only to narrow fields but to bright objects. There are two independent sources of noise in a telescopic image, atmospheric noise and photon noise. The image-improvement system works by feedback of signals from the image to the mirror. The feedback achieves a reduction of atmospheric noise at the cost of an amplification of photon noise. The optimum image-improvement occurs when the two sources of noise become roughly equal. When a bright object is observed in a conventional telescope, the atmospheric noise is enormously greater than the photon noise, and so a large improvement can in principle be obtained by trading off the atmosphere against the photons. When a very faint object is observed, the two sources of noise are comparable, and a tradeoff offers no hope of advantage. Detailed analysis allows us to replace the qualitative words "bright" and "faint" in the foregoing statements by quantitative estimates.

There are two distinct types of image-improvement systems--the single-image type and the multiple-image type. The single-image type is like a conventional telescope, with all the light going to a single image and all the input to the feedback system coming from a single image monitor. The multiple-image type uses an array of secondary mirrors and prisms to split the light from the primary mirror into many beams that are recombined to form many images. Each of the secondary images has its own detector, which acts as an interferometer and feeds back information to the primary mirror. In its extreme form, the multiple-image system has a number of images equal to the number of movable elements in the primary mirror. The extreme multiple-image system makes optimum use of all the information contained in the incident light and gives a final image of the highest possible resolution. This optimum performance will be very costly, requiring large numbers of precise optical and mechanical linkages in addition to the multiple image-monitors and data processors.

The single-image system gives a smaller but still useful image-improvement at much lower cost. Between the single-image and extreme multiple-image systems there are systems of intermediate performance and intermediate cost. Experience will decide how far it is worthwhile to go in pursuit of the ultimate performance. The initial demonstration model of an image-improvement system should be a single-image system of modest aperture. When a working single-image model is available, the next step will be to combine several copies of it into an interferometer array. The array can then be gradually expanded until it becomes a complete multiple-image system.

The results of the theory can be summarized in a single formula for the mean-square phase error of the light waves arriving at the image of a single-image system,

$$D = 0.5\ (p^4 \eta^2 \phi / d^2 \gamma)^{-1/5} \quad . \tag{1.1}$$

Here D is measured in square radians, so that $D < 1$ is a criterion for an acceptable image with diffraction-limited resolution, d is the mirror diameter, γ is the maximum angular frequency of serious atmospheric disturbances, p is the diameter of a mirror whose diffraction limit is equal to the resolution limit defined by the uncorrected atmospheric seeing, η is the "contrastiness" of the image, or the fraction of the light intensity involved in structure inside the seeing limit, and ϕ is the flux of detected incident photons. These quantities are more precisely defined in Eqs. (4.8), (5.37), (5.40), and (10.11) below. The formula (1.1) is to be understood as an order-of-magnitude estimate only. On the right-hand side there is a numerical coefficient that depends on the detailed design of the system. To calculate the coefficient reliably would require a large-scale computer simulation or, even better, an operational system.

The movable elements of the mirror are chosen to have diameter P determined by the seeing conditions for which the system is designed. The optimum choice of P is comparable with the seeing-patch diameter, P. The number of mirror elements contributing to the image is then

$$n = (d^2/P^2) \quad , \tag{1.2}$$

and the criterion for the system to give diffraction-limited resolution is

$$\phi > n\eta^{-2}\,\phi_o \quad , \tag{1.3}$$

$$\phi_o = (\gamma/p^2) \quad . \tag{1.4}$$

The flux ϕ_o depends only on atmospheric seeing conditions and is independent of the system design. As a basis for rough estimates we may take $p = 10$ cm, $\gamma = 10^{-3}\ \mathrm{sec}^{-1}$, so that

$$\phi_o = 10 \text{ photons per cm}^2 \text{ per sec} \quad . \tag{1.5}$$

The flux (1.5) would correspond to an object of visual magnitude 14, if photons could be detected with 100% efficiency.

For a multiple-image system the same formula (1.3) applies, where n is now the number of mirror elements contributing to each image. For the extreme multiple-image system, $n = 2$ independently of the size of the system. If the optical field contains as one of its main features an object with sharply defined structure, such as a star or a star-cluster or the edge of a planet, we can take $\eta = 1$, and Eq. (1.3) reduces to

$$\phi > n\phi_o \quad . \tag{1.6}$$

If the image-detectors have a quantum efficiency of 10%, then Eq. (1.6) implies that any high-contrast object with visual magnitude

$$m \leq 11 \tag{1.7}$$

can be imaged with arbitrarily high resolution in an extreme multiple-image system. To obtain diffraction-limited resolution in a single-image system with aperture d meters, we need a magnitude

$$m \leq 7 - 5 \log_{10} d \quad . \tag{1.8}$$

Single-image systems require bright objects. But the numerical estimates (1.7) and (1.8) should not be taken too seriously. The limiting magnitude that an image-improvement system can handle will vary strongly with the seeing conditions, as indicated by the factor p^4 in Eq. (1.1). At times of particularly good seeing, the limiting magnitude may be considerably fainter than Eq. (1.7) or Eq. (1.8). The true limits can only be discovered by experience with real systems.

The plan of this report is as follows. Section II and Appendix A provide the theoretical basis for the remainder of the report. Section II contains a general prescription for optimizing the performance of any linear servomechanism whose input is subject to quantum fluctuations. An exact expression is obtained for the fluctuations of the output of the optimized servo, given the spectral characteristics of the input. Sections III and IV contain a precise definition of a single-image system and a calculation of its performance. The results of Sections III and IV could be applied to the optimization of the design of an actual system by numerical simulation. But the formulae are too unwieldy for optimization to be carried through in closed analytic form. Sections V and VI contain a more complete analysis of an idealized single-image system, with approximations that need not be made in a computer simulation. The

aim of these sections is to obtain a detailed understanding of the characteristics of an optimized system. The optimization is carried through exactly, and the parameters of the optimum program for feeding back information from image to mirror are completely specified. A particular choice of atmospheric model leads to the result, Eq. (1.1). In Sections VII through X we return to the general single-image system described in Section III, and optimize it using a modified version of the criterion of performance defined in Section IV. We do not make the "big-mirror approximation" on which the explicit optimizations of Section V are based. The optimizations of Section V are carried out in three stages, fixing first the y-dependence of the feedback kernel N, second the x- and t- dependence of N, and third the x-dependence of the tremor-covariance function F. In Section VII we carry through the first two stages without approximation, obtaining analytic expressions for the optimum N corresponding to any given F. These expressions have a simple heuristic interpretation, which is explained in Section VIII. The third stage of optimization, leading to the determination of the optimum F, is described in Section IX. Exact equations defining F implicitly are written down. We introduce simplifying approximations, which are less drastic than the big-mirror approximation of Section V, and still allow F to be calculated explicitly. In this way we recover the results of Section V, under conditions that are realistic enough to be applied to actual telescopes. In particular, the effects of the mirror edges are no longer required to be negligible. The results of Section IX, describing the characteristics and performance of an optimized system, include as special cases both the idealized big-mirror model of Section V and the rigid-mirror pure-tilt model of Section VI. Finally, Section X discusses some points of detail that arise in applying these results to real systems.

The optimization method of Sections VII through IX can be applied almost without change to multiple-image systems, and leads to systems with performance defined by Eq. (1.3) or Eq. (1.6). Multiple-image systems will be analyzed in detail in a subsequent report.

Many questions of great importance in practical telescope design are not discussed in this report. The most essential question left for future investigation is the optimum geometry of an image-improvement system. The theory of Sections III, IV, VII, and VIII applies to a system of arbitrary geometry, with any number of mirrors in an array of any shape, but the optimization analysis of Sections V, VI, IX, and X assumes a conventional single-mirror system. It seems likely that in optical astronomy the relative merits of extended arrays and single dishes will be different for different types of object, as is the case in radio astronomy [for a description of a multiple-mirror optical telescope, see Weymann and Carlton (1972)]. It would be natural to use a single-image improvement system with a single-mirror telescope, or a multiple-image system with a multiple-mirror telescope, but there is no compelling reason for excluding the other two combinations. The combination of multiple-mirror with multiple-image system would allow the most freedom in the choice of geometry. Such a system with a spaced-out array of mirrors would be an interferometer of very high capability. The application of image-improvement to optical long-base-line interferometry looks very promising and deserves a separate analysis.

II THEORY OF A QUANTUM-MECHANICAL SERVOMECHANISM, WITH APPLICATION TO A SIMPLE INTERFEROMETER

We define in abstract mathematical language the situation that is at the heart of any image-improvement system. We have a random variable $a(t)$, not directly observable, whose statistical properties are expressed by the expectation values

$$\langle a(t) \rangle = 0 \quad , \qquad \langle a(s)\, a(t) \rangle = U(s - t) \quad . \tag{2.1}$$

We have a response function $b(t)$, which we are free to program using all the information available to us up to the time t. In the applications, $a(t)$ will be an atmospheric phase shift and $b(t)$ a compensating phase shift applied to a mirror element. Our information comes from a sense organ that detects photons. The probability for a photon to be detected per unit time at time t is

$$I(t) = A + B\, s(t) \quad , \tag{2.2}$$

$$s(t) = a(t) + b(t) \quad . \tag{2.3}$$

The photons arrive at random at times $(t_1, t_2, \ldots)$ following a Poisson distribution with the rate given by Eq. (2.2). The expectation values (2.1) and the coefficients A, B are supposed known. Otherwise, the totality of our information at time t is the series of times of arrival of photons with $t_j < t$.

We suppose that the program defining $b(t)$ is linear in the photon arrival rate. Then the most general program is

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$$b(t) = \sum_j N(t - t_j) - C \quad , \tag{2.4}$$

where C is a constant and $N(\tau)$ is any function of a real variable satisfying

$$N(\tau) = 0 \quad \text{for} \quad \tau < 0 \quad . \tag{2.5}$$

Our objective is to choose the program (2.4) so that b(t) compensates a(t) as well as possible. First, we require zero bias, or

$$\langle s(t) \rangle = \langle b(t) \rangle = 0 \quad , \tag{2.6}$$

which fixes the constant term in Eq. (2.4),

$$C = A \int_0^\infty N(t)\, dt \quad . \tag{2.7}$$

Second, as the criterion for optimization we require that the mean fluctuation

$$D = \langle [s(t)]^2 \rangle \tag{2.8}$$

be as small as possible. The averages in Eqs. (2.6) and (2.8) are taken over both the photon statistics and the random variations of a(t) (atmospheric statistics). We write $\langle \quad \rangle_p$ to denote an average over photon statistics only with a(t) fixed.

We define Fourier transforms by

$$\tilde{U}(\omega) = \int U(t) \exp(i\omega t)\, dt \quad , \tag{2.9}$$

and similarly for $\tilde{N}(\omega)$, $\tilde{a}(\omega)$, $\tilde{s}(\omega)$ and so on. The function $\tilde{U}(\omega)$ is the spectral function of the random variable a(t); it is real, positive, and

even. The function $\tilde{N}(\omega)$ is analytic in the upper half-plane. The results of our analysis are summarized in the following

Theorem: For any feedback program defined by Eqs. (2.4) and (2.7), the mean fluctuation is

$$D = (1/2\pi)\int d\omega \left\{[\tilde{U}(\omega) + A|\tilde{N}(\omega)|^2] / [|1 - B\tilde{N}(\omega)|^2]\right\} \quad . \tag{2.10}$$

The optimum program has N(t) defined by

$$1 - B\tilde{N}(\omega) = \exp\,[(1/2\pi i)\int G(\omega')(\omega' - \omega)^{-1}\, d\omega'] \quad , \tag{2.11}$$

$$G(\omega) = \log[1 + I\tilde{U}(\omega)] \quad , \tag{2.12}$$

$$I = B^2/A \quad , \tag{2.13}$$

and gives the minimum fluctuation

$$D = (1/2\pi I)\int G(\omega)\, d\omega = -\,(A/B)\, N(0+) \quad . \tag{2.14}$$

If an upper bound

$$M = \operatorname*{Max}_{\omega}\,[\omega^2\, \tilde{U}(\omega)] \tag{2.15}$$

exists, then the optimum program gives

$$D < [M/I]^{1/2} \quad . \tag{2.16}$$

The proof of these statements is given in Appendix A. We continue here with a discussion of their physical meaning. The formula (2.10) shows the total system noise D as a sum of two terms, the atmospheric noise proportional to $\tilde{U}(\omega)$, and the photon noise proportional to A, both modified by the feedback factor $|1 - B\tilde{N}(\omega)|^{-2}$. When there is no feedback

$[\tilde{N}(\omega) = 0]$, D reduces to the purely atmospheric noise

$$N_a = (1/2\pi)\int \tilde{U}(\omega)\, d\omega = U(0)$$

$$= \langle [a(t)]^2 \rangle \quad . \tag{2.17}$$

When the feedback is infinitely strong $[\tilde{N}(\omega) \to \infty$ for ω in a finite frequency band of width $\pi\gamma]$, D reduces to the pure photon noise

$$N_p = (A/2\pi B^2)\int d\omega = (\gamma/2I) \quad . \tag{2.18}$$

There is some arbitrariness in the definition of $\pi\gamma$, the bandwidth over which strong feedback is required. For definiteness we choose to define the atmospheric spectrum amplitude a and width γ by

$$a^2 = U(0) \quad , \quad \gamma = (M/2a^2) \quad , \tag{2.19}$$

with M given by Eq. (2.15). Then the optimum program gives, according to Eq. (2.16),

$$D < [2\gamma a^2/I]^{1/2} \quad , \tag{2.20}$$

which may also be written

$$D < 2(N_a N_p)^{1/2} \quad . \tag{2.21}$$

The optimum program thus achieves a tradeoff between atmospheric and photon noise as described qualitatively in Section I.

The motivation for the definition (2.19) is as follows. A standard example of a random process (not necessarily applicable to the real atmosphere) has the autocorrelation function

$$U(t) = a^2 \exp(-\gamma|t|) \quad , \tag{2.22}$$

and the spectrum

$$\tilde{U}(\omega) = [2\gamma a^2/(\omega^2 + \gamma^2)] \quad . \tag{2.23}$$

For this process, γ defined by Eq. (2.19) coincides with the decay-constant γ in $U(t)$. It is easy to compute the optimum program explicitly for the spectrum (2.23). We find

$$B\,N(t) = -(\delta - \gamma)\exp(-\gamma t) \quad , \quad t > 0 \quad , \tag{2.24}$$

$$B\,\tilde{N}(\omega) = -i(\delta - \gamma)(\omega + i\gamma)^{-1} \quad , \quad \mathrm{Im}\,\omega > 0 \quad , \tag{2.25}$$

$$\delta = (\gamma^2 + 2a^2\gamma I)^{1/2} \quad , \tag{2.26}$$

and hence

$$D = [(\delta - \gamma)/I] = (2\gamma a^2/I)^{1/2}[(1 + z)^{1/2} - z^{1/2}] \quad , \tag{2.27}$$

$$z = (\gamma/2a^2 I) = (N_p/N_a) \quad . \tag{2.28}$$

For the interesting case in which feedback is strong, $N_a \gg N_p$, z is small, and the estimate (2.20) of the optimum system noise comes very close to the true value, Eq. (2.27). We may thus expect that Eq. (2.20) provides a reasonably quantitative estimate of the system performance, for any process with a spectrum not radically different in shape from Eq. (2.23).

If the atmospheric fluctuation spectrum has a sharper cutoff at high frequencies, so that the long tails of Eq. (2.23) are absent, then the performance is far better than the estimate (2.20) would indicate. In

this case it is convenient to define another "effective spectrum width" by

$$\pi\bar{\gamma} = \int (I\tilde{U}(\omega)/[1 + I\tilde{U}(\omega)]\, d\omega \quad . \tag{2.29}$$

This $\pi\bar{\gamma}$ comes closer to the true spectrum width of $\tilde{U}(\omega)$ as the cutoff at high frequencies becomes sharper. For the spectrum (2.23), $\bar{\gamma}$ is much larger than γ, and so $\bar{\gamma}$ is an inappropriate quantity to describe the spectrum width. On the basis of the definition (2.29) we can prove that an optimum program gives

$$D < \bar{N}_p (1 - \bar{z})^{-1} \log(\bar{z}^{-1}) \quad , \tag{2.30}$$

$$\bar{z} = (\bar{N}_p/N_a) \quad , \quad \bar{N}_p = (\bar{\gamma}/2I) \quad . \tag{2.31}$$

For the proof see Appendix B. The result, Eq. (2.30), shows that for a spectrum with sharp cutoff the atmospheric fluctuations can be compensated very efficiently, so that the total system noise with the optimum program exceeds the pure photon noise only by a logarithmic factor.

In our discussion of the image-improvement problem we shall make use of the pessimistic estimate, Eq. (2.20) or Eq. (2.27), for D and not the optimistic estimate, Eq. (2.30). In calling these estimates pessimistic and optimistic, we do not imply any doubt concerning their validity. Both are strictly valid in all cases. However, Eq. (2.20) is closer to equality for a long-tailed spectrum, while Eq. (2.30) is closer to equality for a cutoff spectrum. The long-tailed spectrum makes the task of image-improvement harder. Therefore the use of Eq. (2.20) makes our estimates of the system performance conservative.

For completeness we add here two more estimates for the optimum D, each of which may be convenient for spectra of shape intermediate between

long-tailed and cutoff. The inequality (2.20) holds with γ replaced by either γ_1 or γ_2, where γ_1 and γ_2 are two alternative definitions of the spectrum width,

$$\gamma_1 = (c/4\pi)\left[\left\{\int [\tilde{U}(\omega)]^{1/2} d\omega\right\}^2 \Big/ \left|\int \tilde{U}(\omega)\, d\omega\right|\right] \quad , \tag{2.32}$$

$$\gamma_2 = (c/2)\left\{\left[\int \tilde{U}(\omega)\, \omega^2 d\omega\right] \Big/ \left[\int \tilde{U}(\omega)\, d\omega\right]\right\}^{1/2} \quad , \tag{2.33}$$

and

$$c = 0.6476 = \mathrm{Max}_x \left\{ x^{-1} [\log (1 + x)]^2 \right\} \quad . \tag{2.34}$$

The proof of Eq. (2.20) with these definitions is given in Appendix C. The definition (2.33) is (apart from the numerical coefficient) the most natural definition of a spectral width, but unfortunately γ_1 and γ_2 are both infinite for a long-tailed spectrum such as Eq. (2.23).

To conclude this section, we discuss the application of the foregoing theory to the problem of optical phase-compensation in a simple interferometer. Suppose we have an interferometer measuring the resultant of two coherent light signals with amplitudes f, g, and phase-difference

$$s(t) - \varphi \quad . \tag{2.35}$$

Here s(t) is given by Eq. (2.3), a random phase-difference a(t) produced by the atmosphere plus the compensating programmed phase difference b(t). The angle φ may be chosen to make the interferometer as effective as possible in keeping s(t) small. The output of the interferometer is a photon count at the rate

$$I(t) = R + f^2 + g^2 + 2fg \cos [s(t) - \varphi] \quad , \tag{2.36}$$

where R is the background of incoherent photons. We want to approximate this nonlinear response by the linear expression (2.2) with

$$A = R + f^2 + g^2 + 2fg \cos \varphi \quad , \tag{2.37}$$

$$B = 2fg \sin \varphi \quad . \tag{2.38}$$

The linear approximation will be good if the system succeeds in keeping s(t) small. The replacement of sin[s(t)] by s(t) requires simply

$$D \ll 1 \quad . \tag{2.39}$$

The replacement of cos[s(t)] by 1 in the A term requires

$$|fg \cos \varphi| D \ll A \quad . \tag{2.40}$$

Usually Eq. (2.39) will imply Eq. (2.40), but Eq. (2.40) may be violated even when Eq. (2.39) holds, if we try to operate the interferometer close to zero photon rate, with R small, f and g equal, and φ close to π.

According to Eq. (2.14), D depends only on the parameter $I = (B^2/A)$, and D is minimized when I is maximized. From Eqs. (2.37) and (2.38) we find

$$I = [(2fg \sin^2 \varphi)/(x + \cos \varphi)] \quad , \tag{2.41}$$

$$x = [(R + f^2 + g^2)/(2fg)] > 1 \quad . \tag{2.42}$$

The maximum of I is reached at $\varphi = \varphi_0$ where

$$\cos \varphi = -\zeta \quad , \qquad \zeta = x - \sqrt{x^2 - 1} \quad , \tag{2.43}$$

and there I takes the value

$$I_{Max} = 4fg\,\zeta \quad . \tag{2.44}$$

On the other hand, at $\varphi = \pi/2$ the value of I is

$$I_o = 4fg\zeta/(1 + \zeta^2) \quad . \tag{2.45}$$

The advantage of going from $\pi/2$ to φ_o is only a factor $(1 + \zeta^2)$ in I, which produces a factor $(1 + \zeta^2)^{-1/2}$ in D. The advantage is entirely inconsequential except when x is close to 1. Now x can be close to 1 only when R is small and f and g are almost equal, which is precisely the situation in which the condition (2.40) may be violated. In fact the choice $\varphi = \varphi_o$ gives the maximum advantage of a factor $2^{-1/2}$ in D, only when the interferometer is operating in a state of almost complete destructive interference with a very low counting rate. The regime of strong destructive interference is highly unstable against departure of the amplitudes f and g from equality, or against increase in the incoherent background R. So we conclude that the regime $\varphi = \varphi_o$ is not a good place to operate, and the additional factor $2^{-1/2}$ in D is not in practice attainable. The interferometer will operate stably at $\varphi = \pi/2$ and give a value of D close to the attainable minimum. Also the condition (2.40) then disappears. So we shall assume in all the later discussion that interferometers are operated in quadrature, with the interfering signals held at a phase difference of $(\pi/2)$.

The simple interferometer with $\varphi = \pi/2$ gives, by Eq. (2.41),

$$I = 4\,(n_1 n_2/n) \quad , \tag{2.46}$$

where $n_1 = f^2$, $n_2 = g^2$ are the counting rates of the two coherent beams in photons per second, and n is the total counting rate $n = n_1 + n_2 + R$. The estimate (2.20) for D becomes

$$D < [\gamma a^2 n/2n_1 n_2]^{1/2} \quad . \tag{2.47}$$

The condition for satisfactory control of s(t) is $D \ll 1$, which means

$$\mathrm{Min}[(n_1/\gamma) \quad , \quad (n_2/\gamma)] \gg (1 + r)\, a^2 \quad , \tag{2.48}$$

where r is the ratio of background R to signal $(n_1 + n_2)$. Equation (2.48) states that each of the two interfering beams must deliver a number of photons greater than $(1 + r)\, a^2$ within the typical period of fluctuation of the atmosphere. If we take as typical of atmospheric disturbances $\gamma = 10^3\ \mathrm{sec}^{-1}$ and $a = \pi$ radians, and assume a simple interferometer with $r = 0$, the condition (2.48) requires each of the interfering beams to contain at least 10^5 photons per second. The result, Eq. (2.48), is basic to the understanding of the image-improvement problem. It defines a limiting light intensity above which all the sensors of an improvement system must operate if they are to achieve effective control of phase errors.

III DEFINITION OF A FEEDBACK PROGRAM FOR A SINGLE-IMAGE OPTICAL IMPROVEMENT SYSTEM

The remaining sections of this report contain a detailed description and optimization of the single-image optical improvement system whose general nature and purpose were discussed in Section I. A system of the same type has also been described by Muller and Buffington (1974), who have explored its capabilities with computer simulations and are proceeding with the construction of an experimental model.

The basic idea of Muller and Buffington is to measure in real time the quantity

$$S(t) = \int I(y,t)\ I_o(y)\ dy \quad , \tag{3.1}$$

where $I(y,t)$ is the actual brightness at the point y of the image, and $I_o(y)$ is the brightness of an ideal comparison image. They call $S(t)$ the "sharpness" of the image. They prove that, under certain specified conditions, an image that maximizes $S(t)$ also provides a faithful picture of the object. They use $S(t)$ both as the criterion of performance of the system and as the sensory input to their feedback system. Their program is designed to hunt for the maximum of $S(t)$, with no other information than $S(t)$ itself to determine the curve of pursuit. Since the hunt takes place in a multidimensional space of mirror displacements amid a background of rapidly fluctuating atmospheric perturbations, it is important to make the strategy of the hunt as efficient as possible.

The essential difference between the system described in this report and the Muller-Buffington system is that we here separate the assessment function of $S(t)$ from the hunting function. We still use the quantity

S(t), or more accurately the similar quantity defined by Eq. (4.1) below, to assess the performance of the system. But we allow all the information contained in the image-distribution $I(y,t)$, and not only the single number $S(t)$, to be used as sensory input for the hunting program. Our strategy for the hunt is to take the very general feedback program defined by Eq. (3.8), and to optimize it using $S(t)$ as the criterion of success. To use a pictorial analogy, Muller and Buffington are in the situation of a man trying to stay near the top of a multidimensional mountain in terrain wracked by constant earthquakes, with no other sensory data than the readings of a pocket altimeter. We allow the man to use his eyes to obtain a partial view of the terrain surrounding him.

We pay a price for using a much larger sensory input, in that our hunting program, Eq. (3.8), is restricted to be linear in the input data $I(y,t)$. The class of nonlinear programs is too large for us to find the optimum in it by analytical methods. The best we can do is to find the optimum linear program. Muller and Buffington use a nonlinear program to make the best use of their single input $S(t)$. So it is not necessarily true that our optimum program is better than theirs. Both hunting strategies should be tested in operational systems to see which works best. In Section VI of this report we examine briefly the extension of our analysis to include some nonlinear programs.

The starting-point of our analysis is the equation

$$\langle I(q,y)\rangle_p = \int d\theta \; (q/2\pi L)^2 \; O(q,\theta)$$

$$\left|\int dx \exp[iqx \cdot (\theta - y/L) + iq\,\epsilon(x)]\right|^2 \quad , \qquad (3.2)$$

relating the statistical expectation value of the flux $I(q,y)$ of photons of wavenumber q arriving at a point y of the image plane to the flux $O(q,\theta)$ of photons incident from the direction θ in the sky. We take θ

to be a 2-dimensional vector defined by the components of the photon direction normal to the optical axis. L is the focal length of the system and x is integrated over the mirror. The error term $\epsilon(x)$ represents the displacement of the wavefront of a photon reflected from the mirror at the point x. We suppose that $\epsilon(x)$ is a superposition of effects arising from atmospheric inhomogeneities in the light-path and from displacement of the mirror surface. The use of Eq. (3.2) implies two basic approximations. First, there is the geometrical-optics approximation of representing the entire effect of atmospheric refraction by a change of phase of the wave amplitude. Second, there is the narrow-field approximation of replacing the displacement $\epsilon(x,q,\theta)$, which should depend on the photon parameters q and θ, by a function of x alone. The width of field over which image improvement is possible is defined precisely by the range of directions θ over which the θ-dependence of ϵ can be neglected. The neglect of the q-dependence of ϵ means that we are treating the atmosphere as nondispersive.

The formula (3.2) belongs to classical electromagnetic theory. It describes the expected value of the flux of photons, ignoring quantum fluctuations. To take into account the quantum nature of light, we must compute quantities such as

$$K(y,t,y',t') = \langle I(y,t)\ I(y',t')\rangle_p - \langle I(y,t)\rangle_p \langle I(y',t')\rangle_p \quad , \tag{3.3}$$

following the method of Appendix A. In the calculation we take into account the correlation between photon statistics at different times produced by the feedback circuit. The light is treated as a rain of incoherent particles, the number of particles arriving in each patch of the image being a random variable with a Poisson distribution. We

completely ignore the subtler effects of partial coherence in bilinear expressions such as Eq. (3.3)--the effects that were exploited by Hanbury, Brown and Twiss (1958) in the design of their intensity interferometer. It seems unlikely that Hanbury-Brown-Twiss effects could be significant in the present context, but the point should perhaps be investigated.

The wavefront displacement $\epsilon(x,t)$ is assumed to be composed of three parts,

$$\epsilon(x,t) = a(x,t) + b(x,t) + c(x,t) \quad . \tag{3.4}$$

Here, $a(x,t)$ is the unknown atmospheric effect that we desire to compensate, $b(x,t)$ is the mirror displacement (multiplied by 2) which we try to make equal and opposite to $a(x,t)$, and $c(x,t)$ is a high-frequency tremor that we apply to the mirror in order to sense the mirror-image interaction. The function of the tremor is in some respects analogous to that of the tremor of the human eye, which provides the brain with ac feedback signals facilitating the precise control of the movement of the eyeball. The frequencies of the tremor are assumed to be in the range of tens of kilohertz or higher, well separated from atmospheric frequencies. We assume for the tremor movement the form

$$c(x,t) = \sum_j p_j(x)\, c_j(t) \quad , \tag{3.5}$$

where the index j denotes a patch or normal mode of the mirror, and the $c_j(t)$ corresponding to different j are oscillating at well separated frequencies. We suppose the $c_j(t)$ normalized so that

$$\mathrm{Av}[c_j(t)\; c_k(t)] = \delta_{jk} \quad ,$$

when averaged over a time comparable to the atmospheric fluctuation periods. Then Eq. (3.5) implies

$$\mathrm{Av}[c(x,t)\, c(x',t)] = F(x,x') = \sum_j p_j(x)\, p_j(x') \quad , \tag{3.6}$$

and the covariance function $F(x,x')$ defines the statistical properties of the tremor. We are free to choose $F(x,x')$ in any convenient fashion, subject to the limitation that the tremor amplitude should be a small fraction of an optical wavelength. For definiteness we restrict the amplitude by assuming

$$F(x,x) < \lambda^2 \quad , \tag{3.7}$$

where $2\pi\lambda$ is an average photon wavelength.

The feedback from image to mirror is defined by a general bilinear expression

$$b(x,t) = \int_{-\infty}^{t} dt' \iint dy\, dx'$$

$$N(x,x',y,t - t')\, c(x',t')\, [I(y,t') - I_L(y,t')] \quad , \tag{3.8}$$

with

$$I(y,t) = \int dq\, I(q,y,t) \quad , \tag{3.9}$$

and $N(x,x',y,\tau)$ a kernel whose values are programmed into the computer that controls the mirror. In Eq. (3.8) we subtract from the momentary image $I(y,t)$ the long-term average image defined by

$$I_L(y,t) = \int_{-\infty}^{t} \psi(t - t')\, I(y,t')\, dt' \quad , \tag{3.10}$$

where $\psi(\tau)$ is a function extending over many seconds, containing only frequencies much lower than the relevant atmospheric ferquencies. The

normalization condition

$$\int \psi(\tau)\, d\tau = 1 \qquad (3.11)$$

ensures that the feedback b(x,t) responds only to atmospheric fluctuations of the image and not to the constant part of the image. The photon fluctuations in I_L will be negligible in comparison with those in I. The physical meaning of Eq. (3.8) is that the computer calculates the correlation between the mirror tremor at x′ and the image response at y, and uses the result to adjust advantageously the displacement of the mirror at x. Our main problem is to determine the optimum choice of the kernel N. Note that we have I(y,t) and not $\langle I(y,t)\rangle_r$ on the right side of Eq. (3.8), so that the effects of quantum statistics are included in the feedback.

The most reliable way to proceed from this point on would be to carry out an accurate computer simulation of the response of the mirror to an imposed a(x,t) derived from some model of the atmospheric fluctuations, using the nonlinear feedback equation (3.8) without further approximation. Instead of this, since our aim is a qualitative understanding of the system rather than numerical accuracy, we immediately linearize Eq. (3.8) by treating all the displacements c(x,t) and

$$s(x,t) = a(x,t) + b(x,t) \qquad (3.12)$$

as small compared with an optical wavelength. We require

$$|s(x,t)| < \lambda \quad , \qquad (3.13)$$

in addition to Eq. (3.7). This requirement means that the feedback system must be successful in keeping the sum s(x,t) small, even when a(x,t) and b(x,t) are individually large. Our procedure is self-consistent only if in fact Eq. (3.13) is satisfied after the optimized kernel N is

chosen. The criterion of optimization will be determined with this condition in mind.

The linearized form of Eq. (3.8), obtained by expanding the exponential in Eq. (3.2) and using Eq. (3.6), is

$$\langle b(x,t)\rangle_p = \int_{-\infty}^{t} dt' \int dx' \,\Gamma(x,x',t-t')\,\langle s(x',t')\rangle_p \quad . \tag{3.14}$$

Because Eq. (3.2) is used, we can obtain a linear equation only for $\langle b\rangle_p$ and not for b itself. The kernel Γ is given by

$$\Gamma(x,x',\tau) = \int dq\,(q^2/2\pi L)^2 \iint dx''\, d\xi\,[F(x'',\xi) - F(x',\xi)]$$
$$2\,\mathrm{Re}\{M(q,x'-x'')L[x,\xi,q(x'-x''),\tau]\} \tag{3.15}$$

where

$$M(q,z) = \int O(q,\theta)\exp(iq\theta\cdot z)\,d\theta \tag{3.16}$$

is the Fourier transform of the object, and

$$L(x,x',z,\tau) = \int N(x,x',y,\tau)\exp(-iy\cdot z/L)\,dy \tag{3.17}$$

is the transform of the feedback kernel. The subtracted term I_L in Eq. (3.8) has disappeared from Eq. (3.15), because the tremor signals vanish when averaged with $\psi(t-t')$. For Eq. (3.14) to hold, we require in addition a consistency condition

$$\int dq\,(q/2\pi L)^2 q \iiiint dx'\,dx''\,d\xi\,d\tau\;[F(x'',\xi) - F(x',\xi)]$$
$$\mathrm{Im}\{M(q,x'-x'')L[x,\xi,q(x'-x''),\tau]\} = 0 \quad , \tag{3.18}$$

which ensures that zero stimulus $s(x',t')$ produces zero response $b(x,t)$. If we write Eq. (3.14) in symbolic form as

$$\langle b\rangle = \Gamma\langle s\rangle = \Gamma(a + \langle b\rangle) \quad , \tag{3.19}$$

we can write the solution as

$$\langle s\rangle = Qa \quad , \quad Q = [1 - \Gamma]^{-1} \quad . \tag{3.20}$$

In longhand, Eq. (3.20) means

$$\langle s(x,t)\rangle_p = \int_{-\infty}^{t} dt' \int dx' \; Q(x,x',t - t') \; a(x',t') \quad , \tag{3.21}$$

with the kernel Q defined by inverting the operator $(1 - \Gamma)$.

There is a close formal analogy between the feedback system defined by Eq. (3.8) and the interferometer described in Section II. One point that is not obvious is the analogy between the use of a sensing tremor $c(x,t)$ and the choice of phase φ in the interferometer. The image intensity $I(y,t')$ is a sum of contributions of the form

$$\exp\{iq\,[s(x_1) + c(x_1) - s(x_2) - c(x_2)]\} \quad , \tag{3.22}$$

from pairs of points (x_1,x_2) on the mirror. The dc signal $I(y,t')$ with $c(x) = 0$ is analogous to an interferometer output with phase $\varphi = 0$. When we add the tremor $c(x)$ and pick out the ac signal from $I(y,t')$ by forming the expression (3.8), we are replacing Eq. (3.22) a contribution proportional to

$$i\,\exp\{iq\,[s(x_1) - s(x_2)]\} = \exp\{i\,[qs(x_1) - qs(x_2) + \tfrac{\pi}{2}]\} \quad . \tag{3.23}$$

The ac signal is thus analogous to the output of an interferometer with $\varphi = \pi/2$. The consistency condition (3.18) is required in order to make the phase relation $\varphi = \pi/2$ exact. We can therefore expect that all the results of Section II have analogs for the system described in this section.

To complete the definition of the system we need a statistical model of the atmosphere. It is unnecessary to specify the behavior of the atmospheric displacements a(x,t) themselves, which are poorly defined and subject to irrelevant large-scale fluctuations. Only differences between the a(x,t) are observable. We therefore specify the atmosphere by means of the covariance function

$$\langle [a(x,t) - a(x',t')]^2 \rangle = U(x - x', t - t') \quad . \tag{3.24}$$

Very little is known observationally about $U(\xi,\tau)$. Since the high-altitude inhomogeneities are usually drifting with the wind, the space and time dependences of $U(\xi,\tau)$ will in general be strongly correlated. No universal formula for $U(\xi,\tau)$ is likely to be valid on all occasions. Nevertheless it is helpful to have some definite model as a basis for discussion, and for this purpose we use the formula

$$U(\xi,0) = \lambda^2 (\xi/p)^{5/3} \quad , \tag{3.25}$$

deduced by Tatarsky (1968) from a theoretical study of atmospheric turbulence. The length p defines the seeing patch size, or smallest scale of optically disturbing fluctuations. The angular-resolution limit set by the seeing is of the order

$$\alpha = (2\pi\lambda/p) \quad . \tag{3.26}$$

Under conditions of good seeing, α is about 1 second of arc and p lies in the range from 10 to 20 cm.

IV CRITERION FOR OPTIMIZING THE SYSTEM

We use as criterion of overall performance of the system of Section III the quantity

$$S_2 = \frac{1}{2} \iint [\epsilon(x) - \epsilon(x')]^2 \, dx \, dx' \qquad (4.1)$$

which is a measure of the mean-square distortion of the wavefront over the whole mirror. The criterion (4.1), in contrast to Dr. Muller's criterion (3.1), cannot be directly measured by monitoring the image. Muller's choice of (3.1) is constrained by practical necessity, since his system works by monitoring S(t) in real time. For us the criterion (4.1) is better, since it defines more sharply the characteristics that should be maximized in a good image, and we are not here discussing a system in which S is measured and maximized in real time. We are using S_2 only as a tool for the theoretical assessment of the overall performance of a system that is specified independently of S_2. In using S_2 we take a "God's-eye view" of the system, calculating the degree of consonance between the actual wavefront and the perfect wavefront that is invisible to mortal eyes.

We next have to choose a standard of comparison to define how small S_2 should be to give acceptable performance. A suitable criterion is $S_2 \ll S_1$, where S_1 is the expression obtained from Eq. (4.1) by taking the phases $q\epsilon(x)$ to have everywhere the root-mean-square value of 1 radian, with $\epsilon(x)$ and $\epsilon(x')$ uncorrelated when $x \neq x'$. The condition $S_2 \ll S_1$ then means that the phase differences between different points on the mirror, which it is the purpose of the system to control, are in fact controlled. To define S_1 precisely we make the replacement

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$$[\epsilon(x)]^2 \to \lambda^2 \quad , \quad \epsilon(x)\,\epsilon(x') \to 0$$

in Eq. (4.1), and obtain

$$S_1 = \lambda^2 A^2 \tag{4.2}$$

where A is the area of the mirror. The ratio

$$D = (S_2/S_1) \tag{4.3}$$

is our normalized criterion of performance, with $D \ll 1$ for a good diffraction-limited image.

When we substitute $\epsilon(x) = s(x) + c(x)$ in Eq. (4.1), and average over atmospheric and photon statistics, the terms in $s(x)$ and $c(x)$ separate, and we find

$$D = D_S + D_c \tag{4.4}$$

with

$$D_S = (1/S_1)\,\frac{1}{2}\iint dx\, dx'\,\langle [s(x) - s(x')]^2 \rangle \tag{4.5}$$

while D_c reduces by Eqs. (3.6) and (4.3) to

$$D_c = F_o/\lambda^2 \quad , \quad F_o = A^{-1}\int F(x,x)\, dx \quad . \tag{4.6}$$

The condition $D < 1$ thus includes Eq. (3.7) automatically. The main task of the statistical analysis is to compute D_S. For this purpose linear relations such as Eqs. (3.14) and (3.21) are insufficient. We need to grapple with photon-photon correlations expressed by quadratic quantities such as Eq. (3.3). The details of the calculation of D_S are given in Appendix D. The result is

$$D_S = (2S_1)^{-1} \iint dx\, dx' \iint dx_1 dx_2$$

$$\iint_{-\infty}^{t} dt_1 dt_2 [Q(x,x_1,t - t_1) - Q(x',x_1,t - t_1)]$$

$$[Q(x,x_2,t - t_2) - Q(x',x_2,t - t_2)]$$

$$\Big\{ - \frac{1}{2} U(x_1 - x_2, t_1 - t_2)$$

$$+ \iiiint dz\, dz'\, dy dt'\, F(z,z')\, I_o(y,t')$$

$$N(x_1,z,y,t_1 - t')\, N(x_2,z',y,t_2 - t') \Big\} \quad . \qquad (4.7)$$

The meaning of this formidable expression may become clearer if it is compared with the corresponding expression (2.10) for the mean-square fluctuation of a simple linear servosystem in Section II. The two factors Q in Eq. (4.7) are exactly analogous to the factor $|1 - B\tilde{N}(\omega)|^{-2}$ in Eq. (2.10). Both arise by inverting the kernel of the linear feedback equation, Eq. (3.14) or (A-3) respectively. The atmospheric noise term U appears in Eq. (4.7) with the coefficient (-1/2) as compared with the corresponding term $\tilde{U}(\omega)$ in Eq. (2.10), because the definitions (3.24) and (2.1) are slightly different. The photon noise term defined by the last fourfold integral in Eq. (4.7) has the same structure as the term $A|\tilde{N}(\omega)|^2$ in Eq. (2.10), N being in each case the feedback kernel and I_o or A the rate of arrival of photons. Only the factor F in Eq. (4.7) has no analog in Eq. (2.10), since it arises from the mirror tremor system which was absent in the simple servo.

The next problem is to optimize the whole system by minimizing D. The quantities that we are free to vary are the tremor autocorrelation

function $F(z,z')$ and the feedback kernel $N(x,z,y,\tau)$. Both F and N occur in a highly nonlinear way in D, since Q is defined as the inverse of $(1 - \Gamma)$, with Γ given by Eq. (3.15). It might seem almost as hopeless to minimize Eq. (4.7) by disentangling it analytically, as to search for a minimum numerically by trying all functions F and N of two and four variables. Fortunately the analogy with Eq. (2.10) comes to our rescue. In the following section we make an approximation that we call the "Big-Mirror Approximation," consisting essentially in the neglect of effects of the mirror edges. With this approximation, D_S can be reduced to a sum of terms of precisely the form of Eq. (2.10), and so the theorem of Section II can be used to minimize it. The minimization can then be carried through analytically to the end, and exact expressions are obtained for the optimum choice of the functions F and N. We do not claim that the big-mirror approximation is accurate for a telescope of practical size. Its purpose is not to give numerically accurate results, but to give a qualitative understanding of the behavior of a real system. Thus, in designing a feedback system for a real telescope, we should use the exact formula (4.7) rather than the big-mirror approximation to estimate its performance. But the realistic minimization of Eq. (4.7) can be done using the big-mirror forms of F and N as starting point. Allowing F and N to have the shape indicated by the big-mirror approximation, with one or two adjustable parameters to fix the overall strength and range of the feedback, we should be able to compute and minimize Eq. (4.7) numerically, and so arrive at a design close enough to the optimum for practical purposes.

V OPTIMIZATION OF THE SYSTEM IN THE BIG-MIRROR APPROXIMATION

The big-mirror approximation consists in imposing periodic boundary conditions on the wave amplitudes at the edges of the mirror. In reality the edges are free and there are no boundary conditions. By imposing periodic conditions we grossly misrepresent all effects arising from the edges of a real mirror. Since the edge effects may be expected to become less important as the mirror is made larger, we may hope that this approximation becomes in some sense asymptotically exact as the size of the mirror tends to infinity. In any case, the approximation may be useful even if it is not asymptotically exact.

For definiteness we suppose that the mirror is a regular hexagon of diameter d measured from edge to edge. Its area is

$$A = \left(\frac{1}{2}\sqrt{3}\right) d^2 \quad . \tag{5.1}$$

The periodicity is that of a regular triangular lattice with lattice-constant d. Nothing essential in our calculation would be changed if we took the mirror and the periodicity lattice to be square, only a hexagonal mirror looks a little more realistic.

To take advantage of the periodicity, it is convenient to choose the normal modes $p_j(x)$ of the mirror tremor to be standing waves with the same period as the mirror. Each $p_j(x)$ is then a sine or cosine of a scalar product $(v \cdot x)$. We shall always use the letter v to denote a vector of the reciprocal lattice-constant $(4\pi 3^{-1/2} d^{-1})$. For each v other than zero, we have a cosine and a sine mode that we suppose to have equal

intensity of tremor. Then the covariance function defined by Eq. (3.6) becomes

$$F(x,x') = F(x - x') \quad , \tag{5.2}$$

invariant under translation and periodic with the periodicity of the mirror. We can expand F into a Fourier series

$$F(z) = A^{-1} \Sigma_v \tilde{F}(v) \exp(iv \cdot z) \quad , \tag{5.3}$$

and similarly for all other periodic functions. The quantity F_o defined by Eq. (4.6) is now

$$F_o = F(0) = A^{-1} \Sigma_v \tilde{F}(v) \quad . \tag{5.4}$$

The basic optical equation (3.2) takes the form

$$\langle I(q,y)\rangle_p = (L^2A)^{-1} \Sigma_v \, O[q,(v/q) + (y/L)]$$

$$\left|\int dx \exp[iv \cdot x + iq\epsilon(x)]\right|^2 \quad , \tag{5.5}$$

when the wave amplitude at the mirror is made periodic. The ideal image obtained from Eq. (5.5) with $\epsilon(x) = 0$ is

$$I_o(q,L\theta) = AL^{-2} \, O(q,\theta) \quad . \tag{5.6}$$

Since the periodic mirror has no edges, there is no diffraction in the absence of phase errors, and so the ideal image is a perfect copy of the object.

Since all points on the mirror are equivalent, the feedback kernel in Eq. (3.8) will also be assumed translation-invariant and periodic. We expand it into a Fourier series in space and a Fourier integral in time,

$$N(x,x',y,\tau) = N(x - x', y,\tau) = (2\pi A)^{-1} \Sigma_v \int d\omega \tilde{N}(v,y,\omega)$$

$$\exp[iv \cdot (x - x') - i\omega\tau] \quad . \tag{5.7}$$

We calculate the kernel $\Gamma(x,x',\tau)$ as before from Eqs. (3.8) and (5.5). We find that Γ is also translation-invariant and periodic, with Fourier transform given by

$$\tilde{\Gamma}(v,\omega) = \tilde{F}(v) \int \tilde{N}(v,L\,\theta,\omega)\, D_2(\theta,v)\, d\theta \quad , \tag{5.8}$$

where $D_2(\theta,v)$ is a second-difference of the object brightness distribution in the direction θ,

$$D_2\,(\theta,v) = \int q^2 dq\, \{O[q,\theta + (v/q)]$$

$$+ O[q,\theta - (v/q)] - 2O(q,\theta)\} \quad . \tag{5.9}$$

The normalization of $\tilde{\Gamma}(v,\omega)$ is chosen as in Eq. (5.7), so that $\tilde{\Gamma}(v,\omega)$ is precisely the eigenvalue of the kernel $\Gamma(x,x', t - t')$ acting on a plane wave $\exp(iv \cdot x' - i\omega t')$. The inversion (3.20) then gives immediately the Fourier transform of the kernel Q,

$$\tilde{Q}(v,\omega) = [1 - \tilde{\Gamma}(v,\omega)]^{-1} \quad . \tag{5.10}$$

It is easily verified that the consistency condition (3.18) is automatically satisfied when the kernels F and L are translation-invariant.

We are now ready to substitute the periodic kernels into Eq. (4.7), which then happily reduces from a tenfold to a fourfold integral,

$$D_S = (G/2\pi)\, \Sigma_v \int d\omega\, C(v)\, \left|1 - \tilde{\Gamma}(v,\omega)\right|^{-2}$$

$$\{-\frac{1}{2}\, \tilde{U}(v,\omega) + A\tilde{F}(v) \iint dq d\theta$$

$$O(q,\theta)\, \left|\tilde{N}(v, L\,\theta, \omega)\right|^2\} \quad , \tag{5.11}$$

with $\tilde{\Gamma}$ given by Eq. (5.8),

$$G = (\lambda^2 A)^{-1} \quad , \tag{5.12}$$

$$C(v) = 1 \quad \text{for} \quad (v \neq 0) \quad \text{and} \quad C(v) = 0 \quad \text{for} \quad (v = 0) \quad . \tag{5.13}$$

The similarity in structure between Eq. (5.11) and Eq. (2.10) of Section II is now clear, and will be exploited in the minimization of D_S. The factor (-1/2) in Eq. (5.11) looks wrong, but is correct. The Fourier transform of any realistic atmospheric covariance function--for example, Eq. (3.25)--is in fact negative.

We optimize the choice of the kernels $\tilde{F}(v)$ and $\tilde{N}(v,y,\omega)$ in three stages, fixing first the y-dependence of $\tilde{N}$, then the (v,ω)-dependence of $\tilde{N}$ and lastly the v-dependence of $\tilde{F}$. The y-dependence of $\tilde{N}$ is simple, since for any fixed values of (v,ω) the kernel $\tilde{N}(v,y,\omega)$ appears in D_S only in the linear combination of Eq. (5.8) and in the quadratic term of Eq. (5.11). The optimum $\tilde{N}$ must therefore be of the form

$$\tilde{N}(v, L\,\theta, \omega) = \tilde{N}(v,\omega)\, D_2(\theta, v)\, [O(\theta)]^{-1} \quad , \tag{5.14}$$

with

$$O(\theta) = \int dq\, O(q,\theta) \quad . \tag{5.15}$$

The recipe (5.14) agrees with the intuitive notion that an optimum feedback will emphasize those points y of the image that correspond to places of maximum contrast on the object. The second difference $D_2(\theta, v)$ picks out the places of high contrast. When Eq. (5.14) is substituted into Eqs. (5.8) and (5.11), the result is

$$D_S = (G/2\pi)\, \Sigma_v \int d\omega\, C(v) \left|1 - \tilde{F}(v)\Delta(v)\, \tilde{N}(v,\omega)\right|^{-2}$$

$$\left\{-1/2\, \tilde{U}(v,\omega) + A\tilde{F}(v)\Delta(v) \left|\tilde{N}(v,\omega)\right|^2\right\} \quad , \qquad (5.16)$$

with

$$\Delta(v) = \int d\theta \{ [D_2(\theta, v)]^2 / O(\theta) \} \quad . \qquad (5.17)$$

$\Delta(v)$ is a measure of the contrastiness of the object at angular scale not greater than $|v\lambda|$, emphasizing second differences of the brightness distribution.

We are now ready to complete the optimization of $\tilde{N}$. For every fixed v, the ω-integral in Eq. (5.16) is precisely of the form of Eq. (2.10) and can be minimized by choosing $\tilde{N}(v,\omega)$ according to Eq. (2.11). Thus, $\tilde{N}(v,\omega)$ is determined as that function analytic in the half-plane ($\mathrm{Im}\,\omega > 0$) for which

$$\left|1 - \tilde{F}(v)\Delta(v)\, \tilde{N}(v,\omega)\right|^2 = 1 - (2A)^{-1}\, \tilde{F}(v)\Delta(v)\, \tilde{U}(v,\omega) \qquad (5.18)$$

on the real ω-axis. With this choice of $\tilde{N}$, the value of D_S becomes, according to Eq. (2.14),

$$D_S = (G/2\pi)\, A\Sigma_v [\tilde{F}(v)\Delta(v)]^{-1}\, C(v)$$

$$\int \log[1 - (2A)^{-1}\, \tilde{F}(v)\Delta(v)\, \tilde{U}(v,\omega)]\, d\omega \quad . \qquad (5.19)$$

To proceed from this point with the optimization of $\tilde{F}(v)$ for any given $\tilde{U}(v,\omega)$ would be possible but tedious. Instead we shall simplify the algebra by making a physical assumption about the atmospheric fluctuation spectrum. We suppose that the atmospheric fluctuations of spatial wavenumber v decay exponentially in time with the dependence

$$\exp[-\gamma(v)|t_1 - t_2|] \quad , \tag{5.20}$$

where $\gamma(v)$ is an arbitrary function of v. This means that the Fourier transform $\tilde{U}$ has the form

$$\tilde{U}(v,\omega) = -W(v)\; 2\gamma(v)\; \{\omega^2 + [\gamma(v)]^2\}^{-1} \quad , \tag{5.21}$$

normalized so that $[-W(v)]$ is the spatial Fourier transform of the equal-time atmospheric covariance function $U(x_1 - x_2, 0)$. The assumption (5.21) is not completely general, but it is also not over-optimistic. In reality it is likely that the high-frequency components of the atmospheric disturbance, which are the most difficult to compensate, will decrease with ω more rapidly than the long-tailed Lorentz shape (5.21) would indicate.

When $\tilde{U}$ has the form (5.21), the optimum $\tilde{N}$ satisfying Eq. (5.18) is given explicitly by

$$\tilde{N}(v,\omega) = -\frac{1}{2}\, i\, A^{-1}\, W(v)\; g[z(v)]\; [\omega + i\gamma(v)]^{-1} \quad , \tag{5.22}$$

with

$$z(v) = \tilde{F}(v)\Delta(v)W(v)\; [A\gamma(v)]^{-1} \quad , \tag{5.23}$$

$$g(z) = 2z^{-1}\; [(1 + z)^{1/2} - 1] \quad . \tag{5.24}$$

Equation (5.22) states that the feedback that produces displacements with spatial wavenumber v on the mirror is applied with the same exponential

time-dependence (5.20) as the natural decay of atmospheric fluctuations at the same wavenumber. The time-dependence of the feedback is thus completely determined by that of the atmosphere. The space-dependence of the feedback has still to be determined by choosing the optimum $\tilde{F}(v)$ to substitute into Eq. (5.22). For $\tilde{U}$ given by Eq. (5.21), Eq. (5.19) reduces to

$$D_S = \frac{1}{2} G \Sigma_v C(v) W(v) \, g[z(v)] \quad . \tag{5.25}$$

To this must be added

$$D_c = G \Sigma_v \tilde{F}(v) \tag{5.26}$$

according to Eqs. (4.6), (5.4), and (5.12). For any given set of functions $W(v)$, $\gamma(v)$ describing the atmosphere and $\Delta(v)$ describing the contrast-scale of the object, it is not difficult to determine $\tilde{F}(v)$ by minimizing the sum of Eqs. (5.25) and (5.26). With the correction that sums over v omit the term $v = 0$, we may drop the factor $C(v)$ in Eq. (5.25) and obtain

$$D = (2A\lambda^2)^{-1} \Sigma_v W(v)[g + (1/4) z\psi(v)] \quad , \tag{5.27}$$

with

$$\psi(v) = 8A\gamma(v) \, [\Delta(v)]^{-1} \, [W(v)]^{-2} \quad . \tag{5.28}$$

It is convenient to take g as the independent variable, using

$$(1/4) z = g^{-2} (1 - g) \quad , \tag{5.29}$$

so that g runs over the range from 1 to 0 as z runs from 0 to ∞. Then the minimum of D occurs when

$$\psi(v) = g^3 (2 - g)^{-1} \quad , \quad \text{for} \quad \psi(v) \le 1 \quad , \tag{5.30}$$

$$g = 1 \quad , \quad z = 0 \quad , \quad \text{for} \quad \psi(v) \ge 1 \quad . \tag{5.31}$$

The analysis thus gives a natural cutoff to the wavenumbers that it is desirable to compensate by feedback. The critical wavenumber V is defined by

$$\psi(V) = 1 \quad . \tag{5.32}$$

For wavenumbers $|v| > V$, $\tilde{F}(v) = 0$ and there is no feedback. The total number of instrumented modes is then

$$n = (4\pi)^{-1} AV^2 \quad . \tag{5.33}$$

If, instead of using sinusoidal waves to define the modes, we divide the mirror into patches and drive each patch in the appropriate fashion, the optimum number of patches will also be approximately equal to n. If we let each patch have size P as defined by its area $(1/4)\pi P^2$, then the size of patch determined by Eq. (5.33) is

$$P = 4 V^{-1} \quad . \tag{5.34}$$

Since we are supposing the number of instrumented modes to be large, we replace the sum in Eq. (5.27) by an integral. Assuming W(v) and $\psi(v)$ to depend only on the magnitude of v and not on direction, we obtain

$$D = (4\pi\lambda^2)^{-1} \int_0^\infty vW(v)\, dv[g(3 - 2g)/(2 - g)] \quad . \tag{5.35}$$

This integral divides into two pieces according to Eqs. (5.30) and (5.31). In the low-v range it is convenient to take g as variable of integration. Then

$$D = (4\pi\lambda^2)^{-1} \int_V^\infty vW(v)\ dv$$

$$+ (2\pi\lambda^2)^{-1} \int_0^1 dg[(3 - 2g)\ (3 - g)/(2 - g)^2]$$

$$v^2\ W(v)\ (d \log v/d \log \psi) \quad . \tag{5.36}$$

To obtain numerical results we must finally make specific choices for the functions $W(v)$, $\gamma(v)$, and $\Delta(v)$. We suppose the atmospheric covariance function at equal times to be given by a power law

$$U(\xi,0) = \lambda^2 (\xi/p)^\beta \quad , \tag{5.37}$$

of which Tatarsky's hypothesis (3.25) is a special case. We assume $0 < \beta < 2$, so that the Fourier transform of U is negative, and according to Eq. (5.21),

$$v^2 W(v) = c(\beta)\ \lambda^2 (pv)^{-\beta} \quad , \tag{5.38}$$

$$c(\beta) = 2^{\beta+2} \sin(1/2\ \pi\beta)\ [\Gamma(1 + 1/2\ \beta)]^2 \quad . \tag{5.39}$$

We suppose the rate of decay of atmospheric fluctuations to be proportional to their spatial wavenumber; thus,

$$\gamma(v) = \gamma p v \quad , \tag{5.40}$$

where γ is the decay rate corresponding to fluctuations at the seeing-patch size p. The assumption (5.40) will certainly be valid for fluctuations that decay through being convected out of the light-path by the wind. It is plausible that Eq. (5.40) will be approximately valid also for fluctuations decaying by turbulent diffusion. Finally we must choose

$\Delta(v)$, which depends on the detailed structure of the object. For definiteness we make the simplest choice, taking $\Delta(v)$ independent of v. If we assume in the definition (5.17) of $\Delta(v)$ that a certain fraction η of the light from the object has very sharp angular structure while the remainder is very diffuse, then we obtain the estimate

$$\Delta(v) = 6\eta^2 \lambda^{-4} \Phi \quad , \tag{5.41}$$

where Φ is the total flux of detected photons, and the mean wavelength $2\pi\lambda$ is defined by

$$\lambda^{-2} = \Phi^{-1} \iint q^2 \; O(q,\theta) \; dq d\theta \quad . \tag{5.42}$$

We take Eq. (5.41) to be the definition of η, the contrastiness of the object. Putting together Eqs. (5.38), (5.40), and (5.41), we obtain from Eq. (5.28)

$$\psi(v) = N^{-1} \; (pv)^{5+2\beta} \quad , \tag{5.43}$$

$$N = \{3[c(\beta)]^2/4\} \; (p^4 \eta^2 \Phi/A\gamma) \quad . \tag{5.44}$$

We write

$$\delta = (5 + 2\beta)^{-1} \quad , \quad \epsilon = \beta(5 + 2\beta)^{-1} \quad . \tag{5.45}$$

Then the optimum patch size defined by Eqs. (5.32) and (5.34) is

$$P = 4p \; N^{-\delta} \quad . \tag{5.46}$$

This is the patch size mentioned in Eq. (1.2) of Section I. The optimum patch size makes the two terms in Eq. (5.36) of comparable magnitude, the first arising from uncompensated phase errors within the individual patches, and the second from compensated phase errors between different

patches. The minimum D is

$$D = k(\beta)\, N^{-\epsilon} \quad , \tag{5.47}$$

with

$$k(\beta) = [c(\beta)/4\pi] \left\{ \beta^{-1} + 2\beta \int_0^1 (3 - 2g)\,(3 - g)\,(2 - g)^{\epsilon-2} g^{-3\epsilon}\, dg \right\} \quad . \tag{5.48}$$

When $\beta = 5/3$, $\epsilon = 1/5$, then $c(\beta) = 5.6182$, $k(\beta) = 0.9313$. Equation (5.47) can also be written in the form

$$D = f(\beta)\, [p^4 \eta^2 \phi/\gamma d^2]^{-\epsilon} \quad , \tag{5.49}$$

with the coefficient

$$f(\beta) = \left\{ \frac{1}{2} \cdot 3^{1/2}\, [c(\beta)]^2 \right\}^{-\epsilon} k(\beta) \quad . \tag{5.50}$$

When $\beta = 5/3$, then $f(\beta) = 0.4805$, and Eq. (5.49) becomes the equation quoted in the Abstract [Eq. (1.1)]. The fact that the coefficient f is less than unity is unexpected good fortune. We do not know how sensitive f may be to the various simplifying assumptions that we have made.

One possible sensitivity of the coefficients that can easily be tested is their dependence on the atmospheric exponent β. One might suspect that the coefficients at $\beta = 5/3$ are fortuitously small because 5/3 lies close to 2. At $\beta = 2$ the covariance function (5.37) has a singular Fourier transform and $k(\beta) = c(\beta) = 0$. An atmosphere with $\beta = 2$ is physically unreasonable and the theory then becomes meaningless. In fact, however, the coefficients are uniformly small over the whole range

$0 < \beta < 2$ for which the theory is applicable. A table of numerical value is supplied in Appendix E.

We have specified the y-dependence and ω-dependence of the feedback kernel by Eqs. (5.14) and (5.22), but we have not yet looked at the v-dependence of $\tilde{F}$ and $\tilde{N}$. It is the v-dependence that determines the spatial pattern of the feedback on the mirror. According to Eqs. (5.23), (5.28), (5.29), and (5.30),

$$\tilde{F}(v) = \frac{1}{2} W(v)\, g(1 - g)\, (2 - g)^{-1} \quad . \tag{5.51}$$

At long wavelengths (small pv), we have

$$W(v) \sim \lambda^2 p^2 (pv)^{-\beta-2} \quad , \tag{5.52}$$

$$g(v) \sim N^{-1/3} (pv)^{(1/3)(5+2\beta)} \quad , \tag{5.53}$$

by virtue of Eqs. (5.30) and (5.43). Therefore,

$$\tilde{F}(v) \sim \lambda^2 p^2 N^{-1/3} (pv)^{-(1/3)(\beta+1)} \quad . \tag{5.54}$$

This means that the tremor covariance function has the behavior

$$F(x,x') \sim \lambda^2 N^{-1/3} (|x - x'|/p)^{-(1/3)(5-\beta)} \quad , \tag{5.55}$$

for separations $|x - x'|$ large compared with p. The exponent in Eq. (5.55) is (-10/9) for $\beta = 5/3$. This shows that the optimized system must have strong long-range correlation of tremor amplitudes at widely separated points on the mirror.

From Eqs. (5.22) and (5.51) we see that the quantity

$$\int \tilde{N}(v,\omega)\, d\omega \tag{5.56}$$

has the same v-dependence as $\tilde{F}(v)$. The spatial Fourier transform of Eq. (5.56) is the feedback kernel

$$N(x - x' , y , 0) \quad , \tag{5.57}$$

taken with the time of the mirror points x, x' infinitesimally later than the time of the image-point y. This equal-time feedback kernel therefore has the same behavior as $F(x,x')$,

$$N(x - x' , y , 0) \sim (|x - x'|/p)^{-(1/3)(5-\beta)} \quad , \tag{5.58}$$

for pairs of points widely separated on the mirror. It is important that in setting up the feedback program according to Eq. (3.8), we include a long-range interaction feeding the tremor amplitude at x' back into the mirror response at x. If the Tatarsky model atmosphere is realistic, the instantaneous feedback should vary with the inverse (10/9) power of the distance. The feedback at later times will decrease with distance even more slowly.

We have come to the end of the description of our ideal optimized and linearized feedback system. It remains to discuss how relevant this description may be to the requirements of the real nonlinear and non-periodic world.

VI PRACTICAL CONSIDERATIONS, EDGE EFFECTS, AND NONLINEAR SYSTEMS

The most obvious deficiency of the periodic big-mirror approximation is that it does not allow the mirror as a whole to tilt. A tilt of the mirror produces a wavefront displacement $\varepsilon(x)$ that is a linear function of x. When we identify opposite edges of the mirror we exclude the possibility of a linear component, either in the atmospheric wavefront displacement $a(x)$ or in the mirror response $b(x)$.

In the real world, atmospheric disturbances have important linear components that produce displacement of the optical image without distortion. In small telescopes these wandering displacements of the image are the main component of "seeing." The wandering motions can be corrected by a servo-controlled tilt of the mirror. An image-centering system to remove this component of seeing was built and successfully applied to planetary photography by Leighton (1956).

The importance of the mirror edges in a real telescope is shown most clearly when we use as criterion of image quality the expression

$$S' = \iint dq\, dy[I(q,y) - I_o(q,y)]^2 \quad , \tag{6.1}$$

instead of Eq. (4.1). This expression looks at first glance roughly equivalent to Eq. (4.1). When we expand S' to second order in the wavefront displacements, we obtain, instead of Eq. (4.1),

$$S' = \iint dq\, dz(q^2/2\pi L)^2 \left|M(q,z)\right|^2[j(z)]^2 \quad , \tag{6.2}$$

with

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$$j(z) = \int dx\ \epsilon(x)\ [\chi(x - z) - \chi(x + z)] \quad , \tag{6.3}$$

where $\chi(x)$ is defined to be 1 when x is a point on the mirror and 0 otherwise. The expression (6.2) consists entirely of edge effects, since for fixed z the integral (6.3) brings contributions only from points x belonging to two crescent-shaped regions within a distance $|z|$ from the mirror edge. When we make the big-mirror approximation and abolish the edge, $\chi(x) = 1$ everywhere and Eq. (6.2) is identically zero. In the big-mirror approximation the first non-vanishing terms in S' are of fourth degree in $\epsilon(x)$ and cannot conveniently be used for optimization. But the second-degree terms (6.2) are nonzero for a real mirror. The edge effects represented by Eq. (6.2) are nonexistent in the big-mirror approximation but constitute the leading source of image degradation in real telescopes.

To explore the problem of edge effects in a slightly more adequate fashion, we assume that the mirror is very large but no longer periodic. We assume that the mirror diameter d is large compared with all values of z contributing importantly to the integral (6.2). Then all points x contributing to Eq. (6.3) lie close to the mirror edge. We make the further approximation of taking $\epsilon(x)$ in the neighborhood of the edge to be a function only of the azimuthal angle φ of the point x on the mirror. Then Eq. (6.3) becomes

$$j(z) = 2\pi d(z \cdot \epsilon) \quad , \tag{6.4}$$

$$\epsilon = (2\pi)^{-1} \int \epsilon(\varphi)\ n(\varphi)\ d\varphi \quad , \tag{6.5}$$

where $n(\varphi)$ is the unit vector in the direction φ in the plane of the mirror, and ϵ is a vector defining the centroid of the distribution of $\epsilon(x)$ around the mirror circumference. When Eq. (6.4) is substituted into (6.2), the result is

$$S' = \iint dq\,dy \, \left| (a \cdot \text{grad}) \, I_o(q,y) \right|^2 \quad , \tag{6.6}$$

by virtue of Eq. (3.2), where a is the vector

$$a = (2L/d) \, \epsilon \quad . \tag{6.7}$$

But Eq. (6.6) is just the value of S' according to Eq. (6.1) when the image is displaced by the small vector a without distortion. This displacement of the image could be exactly compensated by a rigid tilt of the mirror through an angle

$$\theta = (a/2L) = (\epsilon/d) \quad . \tag{6.8}$$

So we reach the important practical conclusion that, insofar as edge-effects can be attributed to the azimuthal distribution of wavefront displacements $\epsilon(x)$ within a narrow strip around the mirror edge, they can be completely corrected by tilting the mirror. This is possible because only the dipole component (6.5) of the azimuthal distribution of $\epsilon(x)$ contributes to image degradation as measured by the criterion S'. The fact that the dipole or tilt component of the wavefront displacement has a disproportionate effect on optical seeing was also demonstrated in an earlier analysis by Fried (1965).

To summarize the foregoing discussion, we may say that there are two regions of the mirror for which the general analytical machinery of Sections III and IV reduces to a manageably simple form. First there is the interior region, where edge effects can be ignored and the results of the periodic big-mirror approximation apply as described in Section V. Second, there is the extreme edge, where according to Eqs. (6.2), (6.4), and (6.5) the main effects of wavefront distortion can be compensated by a simple tilt of the mirror.

It is an interesting exercise to carry through the analysis of Sections III and IV in detail to find the optimum feedback program for the control of mirror tilt. As in Section V, the analysis simplifies enormously when the particular modes of motion under consideration are isolated from other modes. For the tilt problem we take into account only the two independent linear modes of wavefront displacement. The tremor covariance function and feedback kernel are then restricted to be of the form

$$F(x,x') = \Sigma_{ij}\, F_{ij}\, x_i x'_j \quad , \tag{6.9}$$

$$N(x,x',y,\tau) = \Sigma_{ij}\, N_{ij}(y,\tau)\, x_i x'_j \quad , \tag{6.10}$$

where $i,j = 1,2$ refer to two coordinate axes in the plane of the mirror. The optimum feedback kernel has y-dependence given by

$$N_{ij}(y,\tau) = \Sigma_k\, N_{ik}(\tau)\, [I_o(y)]^{-1}\, [\partial^2 I_o(y)/\partial y_j \partial y_k] \quad . \tag{6.11}$$

The second difference that appeared in Eqs. (5.9) and (5.14) is now replaced by a second derivative. The expression (4.7) for the mean-square phase error reduces to

$$D_S = (G/\text{[illegible]})\, \mathrm{Tr}\Big[\{[1 - \Delta\tilde{N}(\omega)]^{\dagger}\}^{-1}\, C$$

$$[1 - \Delta\tilde{N}(\omega)]^{-1}\, \{\tilde{U}(\omega) + A\, \tilde{N}(\omega)\, \Delta[\tilde{N}(\omega)]^{\dagger}\}\Big] \quad , \tag{6.12}$$

where C and Δ are (2×2) matrices defining the mirror geometry and the distribution of optical contrast in the image, and $\tilde{U}(\omega)$ is the spectrum function of the linear component of the atmospheric wavefront disturbance. Equation (6.12) is minimized in the same way as Eq. (5.16), except that there are now only two modes instead of a large number. The final result

of the minimization is

$$D = f\, N^{-1/3}\, (p/d)^{(1/3)(5-\beta)} \quad , \tag{6.13}$$

where f is a numerical coefficient and N is given by Eq. (5.44). We have here made the same approximations and used the same atmospheric model as in the derivation of Eq. (5.47). Equation (6.13) is exactly the result that we would obtain, apart from an unimportant numerical factor, if we picked out of the sum (5.27) two of the longest-wave modes with wave-number $|v| = 4\pi 3^{-1/2}\, d^{-1}$. So we have reached the comforting conclusion that the tilt modes do not behave differently from the internal modes so far as accuracy of control is concerned. Numerically, the mean-square phase error (6.13) is completely negligible compared with the error (5.47) due to internal modes, for any mirror with $d \gg p$. The analysis of the tilt modes has verified that the edge effects are small compared with internal effects for a large mirror, confirming the naive argument that edge modes should be less important because they are less numerous. The results of the periodic approximation in Section V are thus confirmed as being valid for large mirrors. Of course, in a real system one must make sure to include the tilt modes according to Eqs. (6.9) and (6.10) together with the other modes in the feedback program.

The last problem we have to discuss in this section is the appearance in the feedback kernels (5.14), (5.22), or (6.11) of the object intensity $O(q,\theta)$ or the ideal image $I_o(y)$. Neither $O(q,\theta)$ nor $I_o(y)$ is accurately observable. How then can we use these quantities to define the feedback program? There are two ways around this difficulty. There are many astronomical objects--for example, planets, stars, and bright-nucleus galaxies--for which the intensity distribution $O(q,\theta)$ is known with reasonable accuracy. In these cases we are interested in observing with high resolution the fine details of $O(q,\theta)$, knowing its gross features

quite well. We can then define the feedback program using the best available approximation $\bar{O}(q,\theta)$ to the unknown $O(q,\theta)$. For example, for a planet it would probably be sufficient to use for $\bar{O}(q,\theta)$ a uniform disc of the correct size and spectrum. A feedback system designed to optimize the focusing of the gross features of the object will automatically correct the wavefronts so that the unknown fine details can also be seen clearly.

The second way around the difficulty, if no reliable approximation $\bar{O}(q,\theta)$ exists, is to substitute for $O(q,\theta)$ an average $\bar{I}(q,L\,\theta)$ of the observed image $I(q,L\,\theta)$ over some previous period of time, using an averaging process of the type of Eq. (3.10). A scale factor (AL^{-2}) converts from $\bar{I}(q,L\,\theta)$ to an approximate $O(q,\theta)$ in accordance with Eq. (5.6). The average may be taken over a few seconds or minutes of observation so that the photon fluctuations of $\bar{I}(q,L\,\theta)$ are negligible. The entire feedback system with $\bar{I}$ substituted for $O(q,\theta)$ is no longer linear. It has become a highly nonlinear system, relying on the bootstrap effect of an initially poor image to define a feedback to produce a better image to define a better feedback, and so on. If the bootstrap succeeds, the iteration of the feedback produces an average image $\bar{I}(q,L\,\theta)$ that is close to the true shape of the object, and the performance of the whole system is then adequately described by the linear theory of this report. If the bootstrap fails, $\bar{I}(q,L\,\theta)$ never comes close to $O(q,\theta)$ and the linear theory is irrelevant. To give a theoretical criterion for the success of the bootstrap lies beyond the scope of a linear analysis. An analytical treatment of the nonlinear system using the methods described in this report is unlikely to be feasible. To find out whether the nonlinear system will lock itself into a good-image regime of operation or not, we must probably go to a full-scale computer simulation. The computer simulation could also answer such questions as the following. Would it be advantageous to use initially an image $\bar{I}(q,L\,\theta)$ averaged

over a very short period of time, so that fortuitous moments of good seeing would cause the system to lock into the good-image regime? Can we diagnose a good-image lock-in of the system rapidly enough, and lengthen the time-average in $\bar{I}(q,L\,\theta)$ as soon as lock-in occurs, so that subsequent atmospheric fluctuations or moments of bad seeing do not cause the system to come unlocked? These and many other questions will need to be carefully explored before an optimum design can be chosen for a system using nonlinear feedback.

VII GENERAL OPTIMIZATION OF THE FEEDBACK KERNEL

After the special approximations of the last two sections (big-mirror in Section V, and tilt-only in Section VI) we now return to the problem of optimizing the general single-image system described in Sections III and IV. We shall obtain exact expressions for the feedback program of an optimized system, independent of simplifying assumptions. Specifically, the aim of this section is to determine the feedback kernel $N(x,x',y,t-t')$ appearing in Eq. (3.8) so as to minimize the expression (4.7), which represents the mean-square phase error of the system, with the kernel Q defined by Eqs. (3.15) and (3.20). It is convenient to consider, along with the expression D_S which has the tremor amplitude products replaced by their averages according to Eq. (3.6), the similar expression D' which describes the mean-square phase error before averaging over tremor amplitudes. For the first step in the optimization of N, we make D' a minimum before averaging. The choice of N that makes D' a minimum is independent of the $c(x,t)$, and therefore makes the average D_S a minimum also.

We use some of the notations of Appendix D to obtain a more compact expression for D'. In particular, the kernels $B(Y,X)$ and $N(X,Y)$ are defined by Eqs. (D-12) and (D-13). We introduce the equal-time kernel

$$C(X,X') = \delta(t_X - t_{X'})\,[A\delta(x - x') - 1] \quad . \tag{7.1}$$

Then D' is given by substituting from Eq. (D-23) into (4.5)--namely,

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$$D' = (S_1)^{-1} \iiiint dX dX' dX_1 dX_2$$

$$C(X,X') [(1 - NB)^{-1} (X,X_1)] [(1 - NB)^{-1} (X',X_2)]$$

$$\{-\frac{1}{2} U(X_1 - X_2) + \int dY\, N(X_1,Y)\, N(X_2,Y)\, I_o(Y)\} \quad . \tag{7.2}$$

This can be written

$$D' = (S_1)^{-1} \operatorname{Tr}[(1 - B^+N^+)^{-1} C(1 - NB)^{-1}$$

$$\{-\frac{1}{2} U + N I_o N^+\}] \quad , \tag{7.3}$$

where the trace means a cyclic integration over all space and time variables, the adjoint kernels are defined by

$$N^+(Y,X) = N(X,Y) \quad , \tag{7.4}$$

and the kernel I_o is understood to be diagonal in space and time variables. Not surprisingly, the structure of Eq. (7.3) is identical with that of the corresponding expression (6.12) for the simple tilt-only model of a feedback system.

The condition for D' to be stationary under variation of N is

$$\operatorname{Tr}[(R_1 B^+ + R_2 N I_o)\, \delta N^+] = 0 \tag{7.5}$$

for all δN^+ satisfying

$$\delta N^+ (Y,X) = 0 \quad \text{for} \quad t_X < t_Y \quad , \tag{7.6}$$

where R_1 and R_2 are kernels involving X-coordinates only. As in Section V, we do not try to solve Eq. (7.5) in one step but determine first only the Y-dependence of N. The Y-dependence consistent with Eq. (7.5) is

$$N = KB^{+}(I_o)^{-1} \quad , \quad N^{+} = (I_o)^{-1} BK^{+} \quad , \tag{7.7}$$

where K is a kernel operating on the X-coordinates. The expanded form of Eq. (7.7) is

$$N(X,Y) = \int K(X,X') \, B(Y,X') \, [I_o(Y)]^{-1} \, dX' \quad . \tag{7.8}$$

This means that the feedback from image at Y to mirror at X is obtained by applying the kernel K to the response-function of the image at Y to a mirror-displacement at X'. In Section VIII we give an alternative heuristic derivation of Eq. (7.8) that may make its physical meaning clearer. The kernel $N(x,x',y,t - t')$ in the original feedback formula (3.8) is related to K by

$$\begin{aligned} N(x,x',y,\tau) = \int (q^2/2\pi L)^2 \, dq \\ \int M(q,z) \exp(-iq \, z \cdot y/L) \, dz \\ [K(x,x' + z,\tau) + K(x,x' - z,\tau) - 2K(x,x',\tau)] \quad . \end{aligned} \tag{7.9}$$

Since B is an instantaneous kernel, N will have the correct causal behavior (vanishing for $t_X < t_Y$) if K vanishes for $t_X < t_Y$. Substituting from Eq. (7.7) into Eq. (7.3) we find

$$\begin{aligned} D' = (S_1)^{-1} \mathrm{Tr}\{(1 - T'K^{+})^{-1} C(1 - KT')^{-1} \\ [-1/2 \, U + KT'K^{+}]\} \quad , \end{aligned} \tag{7.10}$$

with

$$T' = B^{\dagger}(I_o)^{-1} B \quad . \tag{7.11}$$

The kernel T' is instantaneous, symmetric, and positive-definite. The definition (D-12) of B can conveniently be written in the form

$$B(y,x) = \int dx' \; B_2(y, x - x') \, [c(x') - c(x)] \quad , \tag{7.12}$$

where

$$B_2(y,u) = (2\pi L)^{-2} \int D_2(y/L, k) \exp(-ik \cdot u) \, dk \quad , \tag{7.13}$$

and $D_2(\theta, k)$ is the second difference of the object brightness distribution defined by Eq. (5.9). Equation (7.11) then becomes

$$T'(x_1, x_2) = \iint dx_1' \, dx_2' \; Z(x_1 - x_1' \; , \; x_2 - x_2')$$

$$[c(x_1') - c(x_1)] \, [c(x_2') - c(x_2)] \quad , \tag{7.14}$$

with

$$Z(u_1, u_2) = \int dy \; [I_o(y)]^{-1} \, B_2(y, u_1) \, B_2(y, u_2) \quad . \tag{7.15}$$

In the analysis up to this point we have refrained from averaging over the tremor amplitudes c(x,t), and we obtained the result (7.9), which is independent of the c(x,t). For the second step in the optimization procedure, determining the (x,t)-dependence of N which is fixed by the kernel K, we begin by performing the tremor averages using Eq. (3.6). Then Eq. (7.10) becomes

$$D_S = (S_1)^{-1} \, \mathrm{Tr}[(1 - TK^{\dagger})^{-1} \, C(1 - KT)^{-1} \, (- \tfrac{1}{2} U + KTK^{\dagger})] \quad , \tag{7.16}$$

with

$$T(x_1,x_2) = \iint dx'_1\, dx'_2\, Z(x_1 - x'_1 \quad , \quad x_2 - x'_2)$$

$$[F(x_1,x_2) - F(x_1,x'_2) - F(x'_1,x_2) + F(x'_1,x'_2)] \quad , \tag{7.17}$$

and Z given by Eq. (7.15). Since T' is symmetric and positive definite, so is T. The kernel (-1/2 U) can be replaced in Eq. (7.16) by

$$V = (1 - P)\ (-1/2\ U)\ (1 - P) \quad , \tag{7.18}$$

where P is the kernel replacing any function by its average over the mirror--namely,

$$P(x,x') = A^{-1} \quad , \tag{7.19}$$

A being the area of the mirror. The replacement (7.18) is allowable since

$$TP = PT = CP = PC = 0 \quad , \tag{7.20}$$

by Eqs. (7.1) and (7.11). The kernel V is positive definite since, by Eqs. (3.24) and (7.18),

$$V(x,x',t - t') = \langle a'(x,t)\ a'(x',t')\rangle \quad , \tag{7.21}$$

$$a'(x,t) = a(x,t) - A^{-1}\int a(\xi,t)\ d\xi \quad . \tag{7.22}$$

Our problem now is to determine the kernel K, subject to the causality condition

$$K(x,x',t - t') = 0 \quad \text{for} \quad t < t' \quad , \tag{7.23}$$

so as to make D_S a minimum.

In Appendix F it is proved that the minimum of D_S is attained when

$$K = -L \quad , \tag{7.24}$$

where L is the unique causal kernel satisfying the integral equation

$$L + L^{\dagger} + LTL^{\dagger} = V \quad . \tag{7.25}$$

The value of the minimum is

$$D_S = (S_1)^{-1} \, \mathrm{Tr}[C(L + L^{\dagger})] \quad . \tag{7.26}$$

If we denote by Tr_x a cyclic integration over the space-coordinates only, Eq. (7.26) can be written

$$D_S = (S_1)^{-1} \, \mathrm{Tr}_x[CL(0+)] \quad , \tag{7.27}$$

where L(0+) means the limit of the kernel $L(x,x',t - t')$ as $t \to t'$ from above. When we use the definition (4.2) of S_1, and add to D_S the term D_c representing the mean-square phase-error due to the tremor, we obtain for the total phase-error,

$$D = (\lambda^2 A^2)^{-1} \, \mathrm{Tr}_x\{C[F + L(0+)]\} \quad . \tag{7.28}$$

The prescription (7.7) together with Eqs. (7.24) and (7.25) gives a complete formal solution to the problem of choosing the optimum feedback kernel to use with a given tremor covariance-function F. The heart of the matter lies in the integral equation (7.25), which can also be written

$$(1 + LT)\,(1 + L^{\dagger}T) = 1 + VT \quad . \tag{7.29}$$

The solution of Eq. (7.25) is equivalent to a factorization of the kernel (1 + VT) into a causal and an anti-causal factor, with the causal factor standing on the left. The existence of such a factorization follows from

an old theorem of Plemelj (1908). The proof of existence given here in Appendix F also provides a constructive procedure by means of which L can be computed.

The analysis of this section reduces to the analysis of Section V if we assume the "big-mirror approximation" to hold. In particular, Eqs. (7.26) and (7.29) are analogous to Eqs. (5.19) and (5.18), respectively. The main difference between the two analyses is that in the big-mirror approximation all the kernels F, C, T, V, L, $L^\dagger$ are simultaneously diagonal in the momentum-representation and commute with each other, whereas in the general case they do not commute. When L, $L^\dagger$, and T commute with each other, the factorization (7.29) can be performed simply by taking logarithms on both sides, and this is the origin of the logarithm in Eq. (5.19). When L and $L^\dagger$ do not commute it is not possible to separate the two factors on the left of (7.29) by using logarithms, and we have to use the integral equation (7.25) instead of an explicit formula for L.

It is a satisfactory feature of our analysis that the optimum feedback program depends on the kernels V (defining the statistical behavior of the atmosphere) and T (defining the optical structure of the image and mirror), but is independent of C (defining the criterion for optimizing the system). The choice of C was arbitrary, and it is reassuring to find that any positive-definite kernel C would do equally well and would lead to the same optimized feedback program. The only requirement that we have imposed on C, apart from positivity, is Eq. (7.20), which states that C is indifferent to constant phases and measures only phase-differences. The special choice (7.1) for C corresponds to the choice of S_2 given by Eq. (4.1) for the criterion of performance of the system. With this choice of C, Eq. (7.28) reduces to the simple form

$$D = (\lambda^2 A)^{-1} \, \mathrm{Tr}_x [F + L(0+)] \quad . \tag{7.30}$$

In deriving Eq. (7.30) we have assumed that F and L satisfy the conditions

$$FP = PF = LP = PL = 0 \quad , \tag{7.31}$$

consistent with the principle that a constant phase-shift is optically irrelevant.

In the analysis of this section we have ignored the supplementary condition (3.18). It is easy to verify that (3.18) is not automatically satisfied when the feedback kernel N has the structure (7.7). But we can always satisfy (3.18) by choosing a kernel K with the property

$$\int K(x,x', \tau) \, d\tau = 0 \quad . \tag{7.32}$$

To make (7.32) hold we have only to subtract from K its long-term average, in the same way as we subtracted I_L from I in Eq. (3.8).

VIII HEURISTIC INTERPRETATION OF THE OPTIMUM FEEDBACK SYSTEM

Experience with the statistical analysis of random events has led us to consider a "chi-squared test" as the most efficient criterion for assessing the significance of deviations of an observed distribution of events from a theoretical model. An optical improvement system must use as its input data the deviations of the observed photon intensity distribution $I(y)$ from the ideal image intensity $I_o(y)$. It is plausible that an efficient measure of significance of the deviation of I from I_o is provided by the quantity "chi-squared,"

$$\chi^2 = \int [I_o(y)]^{-1} [I(y) - I_o(y)]^2 \, dy \quad , \tag{8.1}$$

commonly used in statistical analysis. A reasonable strategy for an optical improvement system would be to control the mirror displacements at each instant in such a way as to follow the line of steepest slope in the direction of diminishing χ^2.

The linearized expression for $I(y)$ in terms of the mirror-displacements $s(x)$ is

$$I(y) - I_o(y) = \int B(y,x) \, s(x) \, dx \quad , \tag{8.2}$$

with B defined by Eq. (D-12). We may therefore write

$$B(y,x) = [\partial I(y)/\partial s(x)] \quad . \tag{8.3}$$

The gradient defining the line of steepest slope of χ^2 in the space of displacements $s(x)$ is

$$\text{grad}\,\chi^2 = [\partial\chi^2/\partial s(x)]$$

$$= 2\int dy\; B(y,x)\; [I_o(y)]^{-1}\; [I(y) - I_o(y)] \quad , \tag{8.4}$$

or in operator notation

$$\text{grad}\,\chi^2 = 2\; B^{\dagger}(I_o)^{-1}\; (I - I_o) \quad . \tag{8.5}$$

The natural definition of a feedback system in which the mirror-displacement b(x) tries to follow the line of steepest decrease of χ^2 would be

$$b = \frac{1}{2}\, K\; \text{grad}\,\chi^2 = K\; B^{\dagger}(I_o)^{-1}\; (I - I_o) \quad , \tag{8.6}$$

where K is a kernel allowing the servo-system to integrate signals over a certain area and over a certain time. The recipe (8.6) is to be compared with the formula

$$b = N(I - I_o) \tag{8.7}$$

defining the feedback kernel N according to Eq. (3.8). The form of N given by Eq. (8.6) is identical with our old result (7.7). So we see that the heuristic method of programming the feedback, based on the gradient of χ^2, gives the same result as the rigorous optimization based on Eq. (4.12). This is a useful check on the consistency of our procedure, verifying incidentally that the treatment of photon statistics in the derivation of Eq. (4.12) was correct.

If we substitute from Eq. (8.2) into (8.1) and use Eq. (7.11), we find

$$\chi^2 = \iint T'(x,x')\; s(x)\; s(x')\; dx\, dx' \quad . \tag{8.8}$$

If the tremor amplitudes are averaged in computing χ^2, then T' is replaced by T. We have, therefore,

$$T(x,x') = \frac{1}{2} [\partial^2 \chi^2 / \partial s(x) \, \partial s(x')] \quad , \tag{8.9}$$

which gives a simple interpretation for the kernel T and explains why T plays such a prominent role in the optimization problem. T in fact defines the shape of the valley around the minimum of χ^2 in the multidimensional space of mirror-displacements s(x). The optical improvement system is designed to hunt as efficiently as possible for the bottom of the valley. The larger and more isotropic the kernel T, the narrower the valley. The mean-square phase error D is thus strongly dependent on T and decreases as T increases.

IX FINAL OPTIMIZATION OF THE SYSTEM

The third and last step in the optimization of the system is to choose the tremor covariance function F to make D given by Eq. (7.32) a minimum. Both here and in the previous optimization of the feedback kernel N, we ignore any practical limitations imposed by discrete structure of the mirror. We consider the mirror not as a finite set of rigidly moving facets but as a continuously deformable surface onto which any smoothly varying tremor and feedback motions can be programmed. This treatment of the mirror as a flexible continuum is consistent with the way a practical system would probably be constructed. (The concept is called the "rubber mirror" by Dr. Richard Muller.) Every deformation of a mirror can be expressed as a sum of an infinite series (3.5), each term in the series being a suitably chosen normal mode (a continuous function on the mirror) multiplied by a corresponding amplitude. We shall shortly prove that the optimized system has zero amplitude of tremor for all except a finite set of modes. It is a satisfactory outcome of the theory that, starting without any assumption of discreteness, it leads automatically to an optimum in which only a finite number of discrete modes need to be instrumented. We can imagine a practical system in which tremor and feedback motions near to the theoretical optimum are produced by a driving program of finite complexity.

To minimize Eq. (7.30), it is convenient to go back a step to (7.16). With the replacements (7.18) for (-1/2 U) and (7.1) for C, Eq. (7.16) implies

$$D = (\lambda^2 A)^{-1} \, \mathrm{Tr}\{F + (1 - GT)\, V[1 - (GT)^{\dagger}] + (GT)T^{-1}(GT)^{\dagger}\} \quad , \tag{9.1}$$

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with G defined by Eq. (F-1). Since this choice of G makes D a minimum for fixed F, we have

$$[\partial D/\partial(GT)] = 0 \quad , \tag{9.2}$$

and therefore the condition for an absolute minimum of D is

$$(dD/dF) = (\partial D/\partial F) = 0 \quad . \tag{9.3}$$

In evaluating $(\partial D/\partial F)$ we hold (GT) constant, so that the term in V contributes nothing, and we are left with the condition

$$\mathrm{Tr}[G(dT)\, G^{\dagger}] = \mathrm{Tr}[dF] \quad , \tag{9.4}$$

for all allowed variations dF of F. It is remarkable that the complicated implicit dependence of G on F does not appear in (9.4). The dependence of T on F is linear and is given explicitly by Eq. (7.17). If we regard the quantity

$$\begin{aligned} W(x_1,x_2) &= W(x_1,x_2,\xi_1,\xi_2) \\ &= \iint dx_1'\, dx_2'\, Z(\xi_1 - x_1', \xi_2 - x_2') \\ &\quad [\delta(x_1-\xi_1) - \delta(x_1-x_1')]\, [\delta(x_2-\xi_2) - \delta(x_2-x_2')] \end{aligned} \tag{9.5}$$

as a kernel in the variables (ξ_1,ξ_2) only, with the variables (x_1,x_2) as parameters, then

$$T = \iint dx_1 dx_2\, W(x_1,x_2)\, F(x_1,x_2) \quad . \tag{9.6}$$

If we define a kernel Y by

$$Y(x_1,x_2) = \mathrm{Tr}[G\, W(x_1,x_2)\, G^{\dagger}] \quad , \tag{9.7}$$

then Eq. (9.4) becomes

$$\mathrm{Tr}_x[(Y - 1)\, dF] = 0 \tag{9.8}$$

for all allowed variations dF.

Which variations dF are allowed? Since F is a real symmetric non-negative kernel, it can be written in the form

$$F(x_1, x_2) = \Sigma_\alpha F_\alpha \varphi_\alpha(x_1)\, \varphi_\alpha(x_2) \quad , \tag{9.9}$$

where $\varphi_\alpha(x)$ is some complete set of normalized orthogonal functions on the mirror. The instrumented modes φ_α have $F_\alpha > 0$, and the uninstrumented modes have $F_\alpha = 0$. Since F satisfies Eq. (7.33), there is one uninstrumented mode that is constant on the mirror and is given the label $\alpha = 0$. Thus,

$$\varphi_o(x) = A^{-1/2} \quad , \quad F_o = 0 \quad . \tag{9.10}$$

Otherwise we impose no special requirements on the φ_α. Since F must remain non-negative, the allowed variations dF are of the form

$$dF(x_1, x_2) = \Sigma_{\alpha\beta} dF_{\alpha\beta}\, \varphi_\alpha(x_1)\, \varphi_\beta(x_2) \quad , \tag{9.11}$$

where $dF_{\alpha\beta} = 0$ unless both $F_\alpha > 0$, and $F_\beta > 0$. The meaning of Eq. (9.8) is then either

$$\iint Y(x, x')\, \varphi_\alpha(x)\, \varphi_\beta(x')\, dx\, dx' = \delta_{\alpha\beta} \quad , \tag{9.12}$$

or

$$F_\alpha = 0 \quad , \quad \text{or} \quad F_\beta = 0 \quad , \tag{9.13}$$

for every pair of indices (α, β).

It is easy to verify from Eqs. (9.5), (9.7), (7.15), and (7.13) that Y is a completely continuous positive-definite kernel with finite trace. Therefore the number n of instrumented modes is finite. In fact, Eqs. (9.12) and (9.13) imply

$$n \leq \mathrm{Tr}_x Y \quad . \tag{9.14}$$

This conclusion is satisfactory as far as it goes, but unfortunately Y depends implicitly on F, and so Eq. (9.14) does not give immediately a numerical estimate of n.

Instead of defining G by Eqs. (F-1) and (7.25), we may use the equation

$$(1 - G^{\dagger}T)(1 - GT) = (1 + VT)^{-1} \quad , \tag{9.15}$$

giving G directly in terms of V and T. The complete solution of the optimization problem then consists in solving Eqs. (9.6), (9.9), (9.12), (9.13), and (9.15) for F, G, and T, given the observational quantities V (describing the atmosphere) and W (defining the optics). In principle the quantities V and W could be measured and the kernels F, G, and T derived from Eqs. (9.6) through (9.15) by numerical computation.

To pursue the optimization further by analytic methods, we make simplifying assumptions concerning the input quantities V and W. We assume that there exists a complete family of normalized orthogonal functions $\varphi_\alpha(x)$ on the mirror in terms of which the kernels $U(x - x', t - t')$ given by Eq. (3.24) and $B_2(y, x - x')$ given by Eq. (7.13) are diagonal for all values of $(t - t')$ and y respectively. Since these kernels are functions of $(x - x')$ only, it is not unreasonable to assume that they commute with one another. Our assumption includes as a special case the "big-mirror approximation" of Section V, which neglected entirely

the effects of mirror edges and took the $\varphi_\alpha(x)$ to be plane waves. A more realistic choice for the $\varphi_\alpha(x)$ is to assume that they are eigenfunctions of the Laplacian

$$\nabla^2 \varphi_\alpha(x) = -d_\alpha \varphi_\alpha(x) \quad , \tag{9.16}$$

with the boundary condition

$$(n \cdot \text{grad}) \, \varphi_\alpha(x) = 0 \tag{9.17}$$

at the mirror edges. For a circular mirror the $\varphi_\alpha(x)$ are then ordinary Bessel functions. The functions defined by Eqs. (9.16) and (9.17) would be a convenient choice for the design of a practical system, and they are reasonably close to being eigenfunctions of any smoothly varying displacement kernel $K(x - x')$ on the mirror. However, we develop the theory for a general family of functions $\varphi_\alpha(x)$, and do not assume that they satisfy Eqs. (9.16) and (9.17). The only special property that we require of the $\varphi_\alpha(x)$ is Eq. (9.10). Thus the projection operator P defined by Eq. (7.19) is

$$P(x,x') = \varphi_0(x) \, \varphi_0(x') \quad , \tag{9.18}$$

and

$$P\varphi_\alpha = 0 \quad \text{for} \quad \alpha \neq 0 \quad . \tag{9.19}$$

The assumption that U is diagonal in the functions $\varphi_\alpha(x)$ implies, by Eqs. (7.18) and (9.19),

$$V(x,x',t - t') = \sum_\alpha V_\alpha(t - t') \, \varphi_\alpha(x) \, \varphi_\alpha(x') \quad , \tag{9.20}$$

with $\alpha = 0$ omitted from the summation. We shall always understand a sum over the index α or β to omit the zero mode unless it is stated otherwise.

As in Section V, we suppose that each atmospheric mode decays exponentially in time at a constant rate, so that

$$V_\alpha(t - t') = V_\alpha \exp(-\gamma_\alpha |t - t'|) \quad , \tag{9.21}$$

and the behavior of the atmosphere is completely specified by the parameters V_α, γ_α.

The assumption that B_2 is diagonal implies

$$B_2(y, x - x') = \Sigma_\alpha b_\alpha(y) \varphi_\alpha(x) \varphi_\alpha(x') \quad , \tag{9.22}$$

(zero mode included), where, by Eq. (7.13),

$$b_\alpha(y) = (2\pi L)^{-2} \int D_2[(y/L), k] |\tilde{\varphi}_\alpha(k)|^2 \, dk \quad , \tag{9.23}$$

$$\tilde{\varphi}_\alpha(k) = \int \varphi_\alpha(x) \exp(-ik \cdot x) \, dx \quad . \tag{9.24}$$

When we substitute from Eq. (9.23) into Eqs. (7.15) and (9.5), we find

$$W = \Sigma_{\alpha\beta} \Delta_{\alpha\beta} \varphi_\alpha(x_1) \varphi_\alpha(\xi_1) \varphi_\beta(x_2) \varphi_\beta(\xi_2) \quad , \tag{9.25}$$

$$\Delta_{\alpha\beta} = \iint h(k, k') (|\tilde{\varphi}_\alpha(k)|^2 - |\tilde{\varphi}_0(k)|^2)$$

$$(|\tilde{\varphi}_\beta(k')|^2 - |\tilde{\varphi}_0(k')|^2) \, dk \, dk' \quad , \tag{9.26}$$

$$h(k, k') = (2\pi L)^{-4} \int D_2[(y/L), k] D_2[(y/L), k'] [I_0(y)]^{-1} dy \quad . \tag{9.27}$$

Equation (9.6) then reduces to

$$T_{\alpha\beta} = \Delta_{\alpha\beta} F_{\alpha\beta} \quad , \tag{9.28}$$

where $T_{\alpha\beta}$, $F_{\alpha\beta}$ are the φ_α-representations of the kernels T and F. Note that the right side of Eq. (9.28) is an ordinary product, not a matrix product. Up to this point we have not assumed that F is diagonal in terms of the same family of functions φ_α that diagonalize U and B_2.

We have to solve Eqs. (9.7), (9.8), (9.15), and (9.28) for F, G, and T. If we take F to be diagonal in the φ_α-representation, then T is diagonal by Eq. (9.28), G is diagonal by Eqs. (9.15) and (9.20), and the nondiagonal part of Eq. (9.8) disappears. We therefore obtain a solution with F given by Eq. (9.9). This means that the tremor motions of the mirror are imposed on the same set of normal modes for which the atmospheric kernel V is diagonal. We write T_α for $T_{\alpha\alpha}$, Δ_α for $\Delta_{\alpha\alpha}$, define G_α by

$$G = \Sigma_\alpha G_\alpha(t - t') \varphi_\alpha(x) \varphi_\alpha(x') \quad , \tag{9.29}$$

and introduce Fourier transforms $\tilde{G}_\alpha$, $\tilde{V}_\alpha$ by

$$\tilde{G}_\alpha(\omega) = \int_0^\infty G_\alpha(\tau) \exp(i\omega\tau)\, d\tau \quad , \tag{9.30}$$

so that $\tilde{G}_\alpha(\omega)$ is analytic in the upper half-plane. Equations (9.12), (9.15), and (9.28) then reduce to either

$$\int |\tilde{G}_\alpha(\omega)|^2 d\omega = 2\pi(\Delta_\alpha)^{-1} \quad , \quad \text{or} \quad F_\alpha = 0 \quad , \tag{9.31}$$

$$|1 - T_\alpha \tilde{G}_\alpha(\omega)|^2 = [1 + T_\alpha \tilde{V}_\alpha(\omega)]^{-1} \quad , \tag{9.32}$$

$$T_\alpha = \Delta_\alpha F_\alpha \quad . \tag{9.33}$$

The value of D given by Eq. (7.30) is

$$D = (\lambda^2 A)^{-1} \Sigma_\alpha [F_\alpha + G_\alpha(0_+)] \tag{9.34}$$

When $V_\alpha(\tau)$ is an arbitrary function, the solution of Eq. (9.32) can be obtained by the method explained in Section II. When $V_\alpha(\tau)$ has the special form (9.13), we find

$$\tilde{V}_\alpha(\omega) = 2V_\alpha \gamma_\alpha [\omega^2 + \gamma_\alpha^2]^{-1} \quad , \tag{9.35}$$

$$\tilde{G}_\alpha(\omega) = iV_\alpha g_\alpha [\omega + i\delta_\alpha]^{-1} \quad , \tag{9.36}$$

$$G_\alpha(\tau) = V_\alpha g_\alpha \exp(-\delta_\alpha \tau) \quad , \quad \tau > 0 \quad , \tag{9.37}$$

with the parameters g_α, δ_α, z_α defined by

$$g_\alpha = 2z_\alpha^{-1} [(1 + z_\alpha)^{1/2} - 1] \quad , \tag{9.38}$$

$$\delta_\alpha = \gamma_\alpha (1 + z_\alpha)^{1/2} \quad , \tag{9.39}$$

$$z_\alpha = (2V_\alpha T_\alpha / \gamma_\alpha) \quad . \tag{9.40}$$

Equations(9.31) and (9.34) then become either

$$[g_\alpha^3/(2 - g_\alpha)] = \psi_\alpha \quad , \quad \text{or} \quad g_\alpha = 1 \quad , \text{ and} \tag{9.41}$$

$$D = (\lambda^2 A)^{-1} \Sigma_\alpha [F_\alpha + V_\alpha g_\alpha] \quad , \tag{9.42}$$

with

$$\psi_\alpha = [2\gamma_\alpha / (\Delta_\alpha V_\alpha^2)] \quad . \tag{9.43}$$

The optimization procedure is now complete. For each normal mode of the mirror we have the input parameters V_α, γ_α, Δ_α. Then ψ_α is given by Eq. (9.43), g_α by (9.41), z_α by (9.38) or by

$$z_\alpha = 4g_\alpha^{-2} (1 - g_\alpha) \quad , \tag{9.44}$$

T_α by (9.40), δ_α by (9.39), and F_α by (9.33). The kernels F, G, T are thereby determined. There is only one important remark to be added. Equation (9.41) has a solution with $0 < g_\alpha < 1$ only when $\psi_\alpha < 1$ or

$$V_\alpha^2 \Delta_\alpha > 2\gamma_\alpha \quad . \tag{9.45}$$

When $\psi_\alpha \geq 1$, there is no stationary value of D for positive F_α, and the minimum of D is attained at $F_\alpha = 0$. The theory instructs us to apply tremor and feedback only to the finite set of modes satisfying Eq. (9.45). The instrumented modes are the modes of largest linear scale, for which V_α is greatest and γ_α least. The uninstrumented modes are those of small scale and high frequ ncy, for which compensation by feedback is impossible because there are not enough photons available. The final formula for the mean-square phase error of the system is

$$D = (\lambda^2 A)^{-1} \Sigma_\alpha V_\alpha g_\alpha [(3 - 2g_\alpha)/(2 - g_\alpha)] \quad , \tag{9.46}$$

where g_α is given by Eq. (9.41), the first alternative applying to the instrumented modes, and the second to the uninstrumented modes.

We have now reproduced the results of Section V up to Eq. (5.31), using assumptions that are more general and more realistic than the big-mirror approximation. To translate from the notations of this section to those of Section V, we have only to substitute

$$\alpha \rightarrow v \quad , \quad V_\alpha \rightarrow \frac{1}{2} W(v) \quad , \quad \Delta_\alpha \rightarrow A^{-1} \Delta(v) \quad , \tag{9.47}$$

the remaining symbols being unchanged. The results of Section V from Eq. (5.35) onward, where we supposed the number of instrumented modes to be so large that sums over modes could be replaced by integrals, hold unchanged in the wider framework of this section. When the number of modes is very large, their statistical properties are the same, whether they are plane waves or belong to any other orthogonal family.

X CONCLUDING REMARKS

We conclude this paper with some remarks concerning two questions that are still open for further investigation. One unsolved problem is to estimate quantitatively the errors that arise when we use the approximate equations of Section IX, based on the assumption that all the kernels $U(x - x', t - t')$ and $B_2(y, x - x')$ commute, in situations where these kernels do not commute. The ideal solution to this problem would be to solve exactly the equations (9.6) through (9.15), which are valid for non-commuting kernels, and compare the results with the approximate prescriptions (9.37) through (9.47). If an exact solution is unattainable, the practical alternative is to choose commuting kernels U' and B_2', which are in some sense "close" to the true kernels U and B_2, and define the parameters of the system using Eqs. (9.37) through (9.47) with the approximate kernels. Then the question becomes, how close U' and B_2' must be to U and B_2, in order that the system optimized for U' and B_2' perform well in the real situation with kernels U and B_2. This question could be answered by calculation, using standard methods of perturbation theory with U', B_2' defining the unperturbed system. In the real world the kernels U and B_2 will vary from night to night or from minute to minute, and will never be exactly known. One will necessarily design the system around some assumed kernels U', B_2' that may be chosen to be commuting. The perturbation-theory analysis will show how wide a range of atmospheric and optical conditions the system so designed should be able to handle. It is reassuring to find that the most dramatic result of the approximate treatment--namely, the conclusion that the tremor kernel F has a finite number of nonzero modes--is also true in the exact theory.

If an image-improvement system of the type here proposed ever becomes operational, one valuable byproduct of its operation will be a mass of precise data concerning the statistical behavior of atmospheric fluctuations in space and time. These data will be easily extracted from recordings of the servo-signals fed by the system into the circuits controlling the mirror surface. No data of comparable accuracy and completeness are at present available.

The second topic of this section is the proper general definition of the quantity η appearing in the fundamental formula (1.1) which is supposed to summarize the results of the entire theory. We called η the "contrastiness" of the object, and defined it by Eq. (5.41). That definition was unsatisfactory because it applied only to the special case when the quantity $\Delta(v)$ was independent of v. We here propose a somewhat more general definition.

When the number of modes is large, we may suppose that each normal mode φ_α is mainly made up of Fourier components with wavenumbers concentrated about some mean value k_α. Then Eq. (9.26) implies

$$\Delta_{\alpha\beta} = (2\pi)^4 \, h(k_\alpha, k_\beta) \quad . \tag{10.1}$$

So we may define a mode-dependent coefficient η_α by

$$\Delta_\alpha = 6 \, \eta_\alpha^2 \, \lambda^{-4} \, (\Phi/A) \tag{10.2}$$

instead of Eq. (5.41), and then Eqs. (9.27) and (10.1) give

$$6 \, \eta_\alpha^2 = (\lambda/L)^4 (A/\Phi) \int \{D_2[(y/L), k_\alpha]\}^2 \, [I_o(y)]^{-1} \, dy \quad . \tag{10.3}$$

We write

$$O_2(\theta, v) = \int q^2 \, dq \, O[q, \theta + (v/q)] \quad , \tag{10.4}$$

so that Eq. (5.9) becomes

$$D_2(\theta, v) = 0_2(\theta, v) + 0_2(\theta, -v) - 2\ 0_2(\theta, 0) \quad , \tag{10.5}$$

and Eq. (5.42) gives

$$(\Phi/\lambda^2) = \int 0_2(\theta, 0)\ d\theta \quad . \tag{10.6}$$

The autocorrelation function $\mu(\varphi)$ of the object brightness distribution is

$$\mu(\varphi - \varphi') = \left[\int 0(\theta + \varphi) 0(\theta + \varphi') d\theta\right] / \left\{\int [0(\theta)]^2\ d\theta\right\} \quad . \tag{10.7}$$

To obtain a simple expression for η_α, we assume that the integral in Eq. (10.3) can be estimated by using the rough approximation

$$\int 0_2[(y/L), v]\ 0_2[(y/L), v']\ [I_o(y)]^{-1}\ dy$$

$$= (L/\lambda)^{-1}\ (\Phi/A)\ \mu[\lambda(v - v')] \quad . \tag{10.8}$$

Then Eq. (10.3) reduces to

$$\eta_\alpha^2 = 1 - (4/3)\ \mu(\lambda k_\alpha) + (1/3)\ \mu(2\lambda k_\alpha) \quad , \tag{10.9}$$

the function μ being normalized so that $\mu(0) = 1$ by virtue of Eq. (10.7). Equation (10.9) is not exact but gives a qualitatively correct picture. It shows η_α to be the fraction of the incident light contained in structures with an angular scale smaller than (λk_α). If we assume the object autocorrelation function to have the special form

$$\mu(u) = \eta^2\ \mu_n(u) + (1 - \eta^2)\ \mu_w(u) \quad , \tag{10.10}$$

with μ_n describing a very narrow and μ_w a very wide structure, then

Eq. (10.9) gives $\eta_\alpha = \eta$ for all α, and Eq. (10.2) reduces to the old definition of η.

For practical estimates of system performance, one may use Eq. (1.1) with η^2 defined by averaging Eq. (10.9) over the n instrumented modes α--namely,

$$\eta^2 = 1 - n^{-1} \Sigma_\alpha \left[(4/3)\mu(\lambda k_\alpha) - (1/3)\mu(2\lambda k_\alpha) \right] \quad . \tag{10.11}$$

When the number of modes is large, the sum may be replaced by an integral, and we obtain

$$\eta^2 = 1 - (3\pi V^2)^{-1} \int \left[4\mu(\lambda k) - \mu(2\lambda k) \right] dk \quad , \tag{10.12}$$

with the integral taken over the circle $|k| < V$, and V defined by Eq. (5.32). Equation (10.12) states that η is roughly equal to the fraction of incident light belonging to structures with angular scale smaller than (λV), which by Eq. (5.34) is just the scale of the uncorrected optical seeing. The practical definition of η is thus Eq. (10.12) with $\mu(\psi)$ given by Eq. (10.7). If Eq. (10.12) is not accurate enough, then one should go back to the exact formula (9.47), with g_α given by (9.42) and η_α defined for each α separately by Eqs. (10.2) and (10.9).

Appendix A

PROOF OF THEOREM

Appendix A

PROOF OF THEOREM

Let the input $a(t)$ be fixed while averages are taken over photon statistics only. We first compute the probability $p(t)$ per unit time for a photon to arrive at time t. Equation (2.2) gives

$$p(t) = A + B \langle s(t) \rangle_p \quad , \tag{A-1}$$

while Eqs. (2.4) and (2.7) give

$$\langle s(t) \rangle_p = a(t) + \int N(t - t') \left[p(t') - A\right] dt' \quad . \tag{A-2}$$

Therefore

$$p(t) - \int BN(t - t')\, p(t')\, dt = \left[1 - B\tilde{N}(0)\right]A + Ba(t) \quad , \tag{A-3}$$

from which the Fourier transform

$$\tilde{p}(\omega) = 2\pi A \delta(\omega) + \left[1 - B\tilde{N}(\omega)\right]^{-1} B\tilde{a}(\omega) \tag{A-4}$$

is determined explicitly.

We next repeat this analysis for the probability $p(t,u)$ per unit time for two distinct photons to arrive at the times t, u. Suppose that $t > u$. Then Eq. (2.2) gives

$$p(t,u) = Ap(u) + Bp(u) \langle s(t) \rangle_{p,u} \quad , \tag{A-5}$$

where the average is taken only over photon histories in which a photon arrived at time u. Equations (2.4) and (2.7) give

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$$p(u)\,\langle s(t)\rangle_{p,u} = p(u)\,\left[a(t) + N(t - u)\right]$$

$$+ \int N(t - t')\left[p(t',u) - Ap(u)\right] dt' \quad . \tag{A-6}$$

Therefore

$$p(t,u) - \int BN(t - t')\, p(t',u)\, dt'$$

$$= \left\{Ba(t) + BN(t - u) + A\left[1 - B\tilde{N}(0)\right]\right\} p(u) \quad . \tag{A-7}$$

We write

$$q(t,u) = p(t,u) - p(t)\, p(u) \quad . \tag{A-8}$$

Then Eqs. (A-3) and (A-7) imply

$$q(t,u) - \int BN(t - t')q(t',u)dt' = BN(t - u)p(u) \quad . \tag{A-9}$$

We cannot solve Eq. (A-9) immediately since it holds only for $t > u$. Instead we apply to both sides the operation

$$1 - \int BN(u - u')\, du'$$

and so obtain

$$q(t,u) - \int BN(t - t')\, q(t', u)\, dt'$$

$$- \int BN(u - u')q(t,u')du' + \iint B^2 N(u - u')N(t - t')q(t', u')$$

$$dt'\, du' = BN(t - u)p(u) - \int B^2 N(u - u')N(t - u')p(u')du' \quad . \tag{A-10}$$

This holds for $t > u$, since $t > u$ implies $t > u'$ inside the integrals. But the left side of Eq. (A-10) is symmetric in t, u, and so is the second term on the right. To make the right side symmetric we need only add the term

$$BN(u - t)\,p(t) \tag{A-11}$$

which is zero for $t > u$. Therefore Eq. (A-10) with the additional term holds for all t, u. We can now take the Fourier transform and obtain the simple result

$$\tilde{q}(\omega,\omega') = \{[1 - B\tilde{N}(\omega)]^{-1}[1 - B\tilde{N}(\omega')]^{-1} - 1\}\,\tilde{p}(\omega + \omega') \quad . \tag{A-12}$$

The last step in the proof of Eq. (2.10) is to compute the quantity

$$J(t,u) = \langle s(t)\,s(u)\rangle_p - \langle s(t)\rangle_p \langle s(u)\rangle_p$$

$$= \langle b(t)\,b(u)\rangle_p - \langle b(t)\rangle_p \langle b(u)\rangle_p \quad . \tag{A-13}$$

Equations (2.4) and (A-8) imply

$$J(t,u) = \iint N(t - t')N(u - u')\,dt'du'\,[q(t',u') + \delta(t' - u')\,p(u')] \quad . \tag{A-14}$$

Taking the Fourier transform and using Eq. (A-12), we deduce

$$\tilde{J}(\omega,\omega') = \tilde{N}(\omega)\,\tilde{N}(\omega')[1 - B\tilde{N}(\omega)]^{-1}[1 - B\tilde{N}(\omega')]^{-1}\,\tilde{p}(\omega + \omega') \quad . \tag{A-15}$$

Now Eq. (A-4) gives

$$\langle \tilde{s}(\omega)\rangle_p = [1 - B\tilde{N}(\omega)]^{-1}\,\tilde{a}(\omega) \quad . \tag{A-16}$$

Therefore Eqs. (A-13) and (A-15) imply

$$\langle \tilde{s}(\omega)\,\tilde{s}(\omega')\rangle_p = [1 - B\tilde{N}(\omega)]^{-1}[1 - B\tilde{N}(\omega')]^{-1}$$

$$[\tilde{a}(\omega)\,\tilde{a}(\omega') + \tilde{N}(\omega)\,\tilde{N}(\omega')\,\tilde{p}(\omega + \omega')] \quad . \tag{A-17}$$

We now take averages over the random variable $a(t)$ using Eqs. (2.1) and (A-4). We have then

$$\langle \tilde{s}(\omega)\,\tilde{s}(\omega')\rangle = 2\pi\delta(\omega + \omega')\,|1 - B\tilde{N}(\omega)|^{-2}$$

$$[\tilde{U}(\omega) + A|\tilde{N}(\omega)|^2] \quad . \tag{A-18}$$

Since

$$D = (2\pi)^{-2}\iint \langle \tilde{s}(\omega)\,\tilde{s}(\omega')\rangle\, d\omega\, d\omega' \quad , \tag{A-19}$$

we have proved Eq. (2.10).

We next turn to Eqs. (2.11) through (2.14). The function

$$\varphi(\omega) = [1 - B\tilde{N}(\omega)]^{-1} \tag{A-20}$$

must be regular in the upper half-plane if the servosystem is to be stable. Let $\varphi_o(\omega)$ be the inverse of the expression on the right side of Eq. (2.11). This is also analytic in the upper half-plane and has boundary values for real ω satisfying

$$|\varphi_o(\omega)|^2 = [1 + I\tilde{U}(\omega)]^{-1} \quad . \tag{A-21}$$

To prove Eq. (2.11) we must show that $\varphi(\omega) = \varphi_o(\omega)$ for the minimum D. We write

$$\varphi_o(\omega) = 1 - f(\omega) \quad , \tag{A-22}$$

$$[\varphi(\omega)/\varphi_o(\omega)] = 1 + g(\omega) \quad , \tag{A-23}$$

so that both f and g are analytic in the upper half-plane and tend to zero at least as fast as ω^{-1} at infinity. Manipulation of Eq. (2.10) using Eqs. (A-20) through (A-23) gives

$$D = (1/2\pi I) \int d\omega \{2 \,\mathrm{Re} f(\omega) + 2 \,\mathrm{Re} [f(\omega) g(\omega)] + |g(\omega)|^2\} \qquad \text{(A-24)}$$

The term in Re(fg) vanishes since $f(\omega)\, g(\omega)$ is analytic and of order ω^{-2} at infinity. Therefore D is a minimum when $g(\omega) = 0$, and Eq. (2.11) is proved.

To determine the minimum value of D, we use the fact that the function

$$f(\omega) + \log[1 - f(\omega)] \qquad \text{(A-25)}$$

is also analytic and of order ω^{-2} at infinity. Therefore

$$D = (1/2\pi I) \int d\omega \{-2 \,\mathrm{Re} \log[1 - f(\omega)]\} \quad . \qquad \text{(A-26)}$$

Using Eqs. (A-22) and (A-21) we deduce the first half of Eq. (2.14). The second half of Eq. (2.14) follows from Eq. (2.11) by the same argument. Finally, Eqs. (2.14) and (2.15) imply

$$D < (1/2\pi I) \int \log[1 + IM\omega^{-2}]\, d\omega \quad , \qquad \text{(A-27)}$$

which is identical with Eq. (2.16).

Appendix B

PROOF OF EQUATION (2.30)

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Appendix B

PROOF OF EQUATION (2.30)

The function

$$f(x) = x^{-1}(1 + x)\log(1 + x)$$

$$= \int_0^1 (d\lambda/\lambda)\,[1 - (1 - \lambda)(1 + \lambda x)^{-1}] \qquad \text{(B-1)}$$

is concave for positive x. For any function $g(\omega)$, we define an averaging process by

$$\mathrm{Av}[g(\omega)] = (\pi\bar{Y})^{-1}\int I\tilde{U}(\omega)[1 + I\tilde{U}(\omega)]^{-1}\,g(\omega)\,d\omega \quad , \qquad \text{(B-2)}$$

with $\bar{Y}$ given by Eq. (2.29). The concavity of x implies

$$\mathrm{Av}\{f[g(\omega)]\} \le f\{\mathrm{Av}[g(\omega)]\} \quad . \qquad \text{(B-3)}$$

Take $g(\omega) = I\tilde{U}(\omega)$ in Eq. (B-3). The left side is $(2ID\,\bar{Y})$ by Eq. (2.14). The right side is $f[(\bar{z})^{-1}-1]$, with $\bar{z}$ defined by Eq. (2.31). The inequality (2.30) is thus identical with Eq. (B-3).

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Appendix C

PROOF OF EQUATION (2.20)

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PROOF OF EQUATION (2.20)

We are to prove Eq. (2.20) with γ replaced by γ_1 or γ_2 defined by Eq. (2.32) or (2.33). In the case of γ_1, Eq. (2.20) follows immediately from Eq. (2.14) and the existence of the upper bound (2.34). To deduce the result for γ_2, we use Cauchy's inequality

$$\left\{\int [\tilde{U}(\omega)]^{1/2} d\omega\right\}^2 \leq \left[\int (a + b\omega^2)^{-1} d\omega\right]\left[\int (a + b\omega^2)\, \tilde{U}(\omega)\, d\omega\right]$$

$$= \pi(ab)^{-1/2}\left[a\int \tilde{U}(\omega)\, d\omega + b\int \tilde{U}(\omega)\, \omega^2 d\omega\right] \quad , \qquad \text{(C-1)}$$

valid for all positive a, b. Taking the minimum on the right side, we deduce

$$\int [\tilde{U}(\omega)]^{1/2} d\omega \leq (2\pi)^{1/2}\left[\int \tilde{U}(\omega) d\omega\right]^{1/4}\left[\int \tilde{U}(\omega)\, \omega^2 d\omega\right]^{1/4} \quad , \qquad \text{(C-2)}$$

that is to say, $\gamma_1 \leq \gamma_2$. So the result for γ_2 is also proved.

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Appendix D

CALCULATION OF PHOTON-PHOTON CORRELATIONS

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CALCULATION OF PHOTON-PHOTON CORRELATIONS

The calculations in this appendix are closely parallel to those of Appendix A. The similarity is obscured by the multiplicity of variables and integrations in expressions such as Eqs. (3.15) and (4.7). The essential first step is therefore to introduce a compact set of notations so that the logical structure of the calculation becomes clear.

By X we denote the space and time coordinates (x, t_X) of a point on the mirror. By Y we denote the coordinates (y, t_Y) of a point of the image. A function of one point X or Y we call a vector. Vectors are denoted by

$$A_1 = a(X_1) \quad , \quad A_2 = a(X_2) \quad , \quad \text{etc.} \tag{D-1}$$

$$S_1 = \langle s(X_1) \rangle_p \quad , \tag{D-2}$$

$$I_1 = \langle I(Y_1) \rangle_p \quad , \tag{D-3}$$

$$I_{01} = I_o(Y_1) \quad . \tag{D-4}$$

A function of two points X or Y is called a bivector. Bivectors are

$$S_{12} = \langle s(X_1)\ s(X_2) \rangle_p \quad , \tag{D-5}$$

$$M_{12} = \langle s(X_1)\ I(Y_2) \rangle_p \quad , \tag{D-6}$$

$$I_{12} = \langle I(Y_1)\ I(Y_2) \rangle_p \quad , \tag{D-7}$$

$$\Delta_{12} = \delta(y_1 - y_2)\,\delta(t_{Y1} - t_{Y2})\,\langle I(Y_1)\rangle_p \quad , \tag{D-8}$$

$$J_{12} = S_{12} - S_1 S_2 \quad , \tag{D-9}$$

$$Z_{12} = M_{12} - S_1 I_2 \quad , \tag{D-10}$$

$$K_{12} = I_{12} - I_1 I_2 \quad , \tag{D-11}$$

the last equation being short-hand for Eq. (3.3). Finally we have two kernels B and N defined by

$$B(Y,X) = \int (q^2/2\pi L)^2 dq \int dx' [c(x',t_X) - c(x,t_X)]$$

$$2\mathrm{Re}\{M(q,x - x')\exp[-iq(x - x') \cdot y/L]\}\,\delta(t_Y - t_X) \quad , \tag{D-12}$$

$$N(X,Y) = \iint dx'\, dt'\, N(x,x',y,t_X - t')$$

$$c(x',t')[\delta(t' - t_Y) - \psi(t' - t_Y)] \quad . \tag{D-13}$$

We abbreviate these further by writing

$$B_1 S_1 = \int dX\, B(Y_1,X)\,\langle s(X)\rangle_p \quad , \tag{D-14}$$

$$B_1 S_{12} = \int dX\, B(Y_1,X)\,\langle s(X)\, s(X_2)\rangle_p \quad , \tag{D-15}$$

$$N_2 M_{12} = \int dY\, N(X_2,Y)\,\langle s(X_1)\, I(Y)\rangle_p \quad , \tag{D-16}$$

and so on. The kernel B describes the optical coupling from mirror to image, while N describes the feedback coupling from image to mirror. Until the very end of the calculation we take the atmospheric input

a(X) to be given and average only over the photon statistics. The vector equations are

$$I_1 = I_{01} + B_1 S_1 \quad , \tag{D-17}$$

$$S_1 = A_1 + N_1 I_1 \quad , \tag{D-18}$$

these being respectively the linearized form of Eqs. (3.2) and (3.8). Because the subtracted long-term average term in Eq. (D-13) satisfies Eq. (3.11), we have

$$N_1 I_{01} = 0 \quad , \tag{D-19}$$

and therefore Eqs. (D-17) and (D-18) give

$$S_1 = A_1 + N_1 B_1 S_1 \quad . \tag{D-20}$$

Equation (D-20) becomes identical with Eq. (3.14) after averaging over the tremor amplitudes c(x,t) and dropping the ψ term. The bivector equations are

$$I_{12} = I_{01} I_2 + B_1 M_{12} \quad , \tag{D-21}$$

$$S_{12} = S_1 A_2 + N_2 M_{12} \quad , \tag{D-22}$$

$$M_{12} = A_1 I_2 + N_1 I_{12} + N_1 \Delta_{12} \quad . \tag{D-23}$$

These hold only for $t_1 > t_2$. For example, Eq. (D-21) states that the image at time t_1 is related by the kernel B to the mirror at time t_1, irrespective of the state of the image at a previous time t_2. The relation would be untrue for $t_1 < t_2$, since then the image at t_2 would be

disturbed by feedback from the events at t_1. The other essentially new feature in the bivector equations is the last term of Eq. (D-23). Equation (D-23) states that the mirror at time t_1 is related by the feedback kernel N to the image at all earlier times t_Y, if one also knows the state of the image at another previous time t_2. The term $N_1 I_{12}$ describes the feedback from all times $t_Y \neq t_2$. The term $N_1 \Delta_{12}$ is the additional correlation between mirror at t_1 and image at t_2 produced by direct feedback from t_2 to t_1.

The bivector equations become even simpler when expressed in terms of the reduced bivectors (D-9) through (D-11)--namely,

$$K_{12} = B_1 Z_{12} \quad , \tag{D-24}$$

$$J_{12} = N_2 Z_{12} \quad , \tag{D-25}$$

$$Z_{12} = N_1 (K_{12} + \Delta_{12}) \quad , \tag{D-26}$$

still with the restriction $t_1 > t_2$. It is permissible to substitute from Eq. (D-26) into Eq. (D-24) or Eq. (D-25) without violating the time-sequence, but it is not permissible to substitute Eq. (D-24) into Eq. (D-26). We thus obtain a closed equation for K_{12} only,

$$K_{12} = B_1 N_1 (K_{12} + \Delta_{12}) \quad , \tag{D-27}$$

with J_{12} to be derived from

$$J_{12} = N_1 N_2 (K_{12} + \Delta_{12}) \quad . \tag{D-28}$$

Equation (D-28) is symmetric between the points X_1 and X_2, and therefore holds regardless of the time-order. To obtain a symmetric equation for K_{12}, we operate on Eq. (D-27) with the kernel $(1 - B_2 N_2)$ which commutes

with B_1N_1. This gives

$$(1 - B_1N_1)(1 - B_2N_2)\, K_{12} = B_1N_1\Delta_{12} - B_1N_1B_2N_2\Delta_{12} \quad , \tag{D-29}$$

still for $t_1 > t_2$ only. Now all terms in Eq. (D-29) are symmetric except $B_1N_1\Delta_{12}$. We add to the right side the term $B_2N_2\Delta_{12}$ which is zero for $t_1 > t_2$. The result is symmetric and must therefore hold for all t_1, t_2. We have thus proved

$$(1 - B_1N_1)\,(1 - B_2N_2)\,(K_{12} + \Delta_{12}) = \Delta_{12} . \tag{D-30}$$

Now using Eq. (D-28) and the identity

$$(1 - N_1B_1)\, N_1 = N_1(1 - B_1N_1) \quad , \tag{D-31}$$

we deduce

$$(1 - N_1B_1)\,(1 - N_2B_2)\, J_{12} = N_1N_2\Delta_{12} \quad . \tag{D-32}$$

Combining this with Eqs. (D-9) and (D-20), we obtain

$$(1 - N_1B_1)\,(1 - N_2B_2)\, S_{12} = A_1A_2 + N_1N_2\Delta_{12} \quad . \tag{D-33}$$

In the whole calculation up to this point it was essential to refrain from averaging over the tremor amplitudes $\epsilon(x,t)$ that occur in the kernels B and N. The products BN that appear in Eq. (D-30) are in the wrong order for averaging, since they contain tremor amplitudes referring to different times and so Eq. (3.6) cannot be used. Only in the final result (D-33) the products appear in the order NB. We can now do the tremor averaging by Eq. (3.6). We drop the subtracted ψ-term from N since it becomes negligible after the tremor averaging. We have then

$$NB = \Gamma \quad , \tag{D-34}$$

with Γ defined by Eq. (3.15), and Eq. (D-33) has the solution

$$S_{12} = Q_1 Q_2 [A_1 A_2 + N_1 N_2 \Delta_{12}] \quad , \tag{D-35}$$

with the kernel Q defined by Eq. (3.20). Tremor averaging can be done in the product $N_1 N_2 \Delta_{12}$ since Δ_{12} makes the times t_1 and t_2 equal.

To evaluate D according to Eq. (4.5), we need the quantity

$$\langle [s(x_1,t) - s(x_2,t)]^2 \rangle_p$$

$$= S_{11} - 2\,S_{12} + S_{22}$$

$$= (Q_1 - Q_2)\,(Q_1 - Q_2)\,[A_1 A_2 + N_1 N_2 \Delta_{12}] \quad , \tag{D-36}$$

allowing ourselves a slight abuse of notation in the last line. Now let C_1 denote the constant vector

$$C(X_1) = 1 \quad . \tag{D-37}$$

A constant phase shift produces no effect on the image; thus

$$\Gamma_1 C_1 = B_1 C_1 = 0 \quad , \tag{D-38}$$

as can be verified immediately from Eq. (D-12). Equation (D-38) implies

$$Q_1 C_1 = C_1 \quad , \tag{D-39}$$

and therefore

$$Q_1 C_1 - Q_2 C_1 = C_1 - C_2 = 0 \quad . \tag{D-40}$$

Therefore we may add inside the square bracket in Eq. (D-36) the terms

$$-\frac{1}{2}\left(A_1^2 + A_2^2\right) = -\frac{1}{2}\left(A_1^2 C_2 + C_1 A_2^2\right) \quad , \tag{D-41}$$

the added terms giving zero when operated on by the kernel $(Q_1 - Q_2)$ $(Q_1 - Q_2)$. We have then

$$\langle [s(x_1,t) - s(x_2,t)]^2 \rangle_p$$

$$= (Q_1 - Q_2)(Q_1 - Q_2)\left[-\frac{1}{2}(A_1 - A_2)^2 + N_1 N_2 \Delta_{12}\right] \quad . \tag{D-42}$$

We finally substitute Eq. (D-42) into Eq. (4.5), expand the abbreviated notations, and perform the average over atmospheric statistics using Eqs. (3.24), (D-8), and (D-17). The result is Eq. (4.7).

Appendix E

TABLE OF COEFFICIENTS $c(\beta)$, $k(\beta)$, $f(\beta)$ APPEARING IN EQUATIONS (5.38), (5.47) AND (5.49)

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TABLE OF COEFFICIENTS $c(\beta)$, $k(\beta)$, $f(\beta)$
APPEARING IN EQUATIONS (5.38), (5.47) AND (5.49)

β	ϵ	$c(\beta)$	$k(\beta)$	$f(\beta)$
0	0	0	0.5	0.5
0.2	0.0370	1.2851	0.6064	0.5984
0.4	0.0690	2.6151	0.7255	0.6418
0.6	0.0968	3.9505	0.8532	0.6632
0.8	0.1212	5.2147	0.9808	0.6688
1.0	0.1429	6.2832	1.0929	0.6598
1.2	0.1622	6.9777	1.1649	0.6350
1.4	0.1795	7.0502	1.1591	0.5900
1.6	0.1951	6.1824	1.0200	0.5153
1.8	0.2093	3.9817	0.6692	0.3867
2	0.2222	0	0	0

The maximum value of $k(\beta)$ is 1.1747 at $\beta = 1.30$. The maximum of $f(\beta)$ is 0.6689 at $\beta = 0.78$.

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Appendix F

PROOF OF EQUATIONS (7.24), (7.25), AND (7.26)

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PROOF OF EQUATIONS (7.24), (7.25), AND (7.26)

The proof is divided into two parts. First, we assume that L exists satisfying Eq. (7.25), and prove Eqs. (7.24) and (7.26). Second, we prove the existence and uniqueness of L. The first part of the proof is similar to the proof of Eqs. (2.11) and (2.14) in Appendix A. Only the details are more delicate because we have here to deal with non-commuting kernels. It is no accident that the second equality (2.14) has precisely the same structure as Eq. (7.27).

Given that L exists, we define two causal kernels G and H by

$$G = (1 + LT)^{-1} L \quad , \tag{F-1}$$

$$H = (1 - KT)^{-1} (L + K) T \quad . \tag{F-2}$$

Then

$$(1 - KT)^{-1} = (1 + H)(1 - GT) \quad , \tag{F-3}$$

and Eq. (7.25) becomes

$$(1 - GT)(V + T^{-1})(1 - TG^{\dagger}) = T^{-1} \quad . \tag{F-4}$$

Now D_S is given by Eq. (7.16), with V replacing (-1/2 U) according to Eq. (7.18). Substituting from Eqs. (F-3) and (F-4) into (7.16), we find

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$$D_S = (S_1)^{-1} \, \mathrm{Tr}\{C[(1 + H)\, T^{-1}\, (1 + H^{\dagger})$$

$$+ (1 - KT)^{-1}\, (KTK^{\dagger} - T^{-1})\, (1 - TK^{\dagger})^{-1}]\}$$

$$= (S_1)^{-1} \, \mathrm{TR}\{C[HT^{-1}H^{\dagger} + G + G^{\dagger} + HG + G^{\dagger}H^{\dagger}]\} \quad . \qquad \text{(F-5)}$$

The last two terms in (F-5) give zero trace, since the convolution of two causal kernels is a kernel that is continuous and therefore vanishes at equal times. The term quadratic in H is non-negative since C and T are positive-definite. Therefore D_S attains its minimum value when $H = 0$, and Eq. (7.24) is proved. The minimum is

$$D_S = (S_1)^{-1} \, \mathrm{Tr}[C(G + G^{\dagger})] \quad . \qquad \text{(F-6)}$$

By Eq. (F-1), the difference

$$L - G = LTG \qquad \text{(F-7)}$$

is a convolution of two causal kernels and therefore has zero trace. Hence, Eq. (F-6) implies Eq. (7.26), and the first part of the proof is complete.

We next prove the uniqueness of L. If two causal kernels L_1, L_2 satisfy Eq. (7.25), then by (7.29),

$$(1 + L_2 T)^{-1}\, (1 + L_1 T) = (1 + L_2^{\dagger} T)\, (1 + L_1^{\dagger} T)^{-1} \quad . \qquad \text{(F-8)}$$

But the left side of Eq. (F-8) is causal, and the right side is anti-causal. Therefore both sides must equal 1, and the uniqueness is proved. To prove the existence of L, we follow a simple modification of the argument used by Newton and Jost (1955) to solve the multichannel inverse scattering problem. The Newton-Jost analysis is a generalization

to noncommuting potentials of the original inverse scattering theory of Gelfand and Levitan (1951). We consider the equation

$$G^{\dagger}(t) + \int_{-\infty}^{0} V(t - t')\, TG^{\dagger}(t')\, dt' = V(t') \tag{F-9}$$

as an integral equation for the kernel $G^{\dagger}(x,x',t)$ on the half-space $t < 0$. This is an equation of Fredholm type [see Riesz and Sz.-Nagy (1955), pages 161 to 190]. The kernel $V(x,x',t - t')$, with x and x' restricted to a finite mirror, is completely continuous. Therefore the Fredholm alternative holds; either Eq. (F-9) has a solution, or the corresponding homogeneous equation with zero on the right side has a nontrivial solution. But any solution $f(t)$ of the homogeneous equation would satisfy

$$\int_{-\infty}^{0} f^{T}(t)\, Tf(t)\, dt + \iint_{-\infty}^{0} f^{T}(t) T\, V(t - t') T f(t')\, dt\, dt' = 0 \quad , \tag{F-10}$$

which is impossible since both T and V are positive-definite. So the existence of $G^{\dagger}(t)$ satisfying Eq. (F-9) for $t < 0$ is proved. If we extend the definition by writing $G^{\dagger}(t) = 0$ for $t > 0$, then $G^{\dagger}$ is an anticausal kernel. Since Eq. (F-9) holds for $t < 0$, the kernel

$$L = V - G^{\dagger} - VTG^{\dagger} \tag{F-11}$$

is causal. We define L by Eq. (F-11), which may also be written

$$1 + VT = (1 + LT)\,(1 - G^{\dagger}T)^{-1} \quad . \tag{F-12}$$

The adjoint of Eq. (F-12) implies

$$1 + VT = (1 - GT)^{-1}\,(1 + L^{\dagger}T) \quad . \tag{F-13}$$

By the same argument that we used in proving uniqueness, Eqs. (F-12) and (F-13) imply

$$(1 - GT)(1 + LT) = (1 + L^{\dagger}T)(1 - G^{\dagger}T) = 1 \quad . \qquad \text{(F-14)}$$

Equations (F-12) and (F-14) together imply (7.29), showing that L defined by (F-9) and (F-11) gives the desired solution of (7.25). Also, Eq. (F-14) shows that $G^{\dagger}$ defined by (F-9) is the adjoint of G defined by (F-1). This completes the second part of the proof.

It is of interest to observe that the integral equations that determine the optimum feedback kernel--namely, Eqs. (7.25) and (F-9)--have the same formal structure as the nonlinear and linear integral equations in the Gelfand-Levitan (1951) theory of the inverse scattering problem. The nonlinear equation is Eq. (I) on page 319, and the linear equation is Eq. (II) on page 321 of the Gelfand-Levitan paper. The correspondence between the Gelfand-Levitan kernels and ours is the following. Spectral function $f \rightarrow$ atmospheric kernel V. Transformation kernel $K_1 \rightarrow -L$, and transformation kernel $K \rightarrow -G$. Space-coordinate $x \rightarrow$ time-coordinate t. An even closer correspondence exists between our equations and those of the Marchenko (1955) formulation of the inverse scattering problem. Our Eqs. (7.25) and (F-9) correspond to Marchenko's Eqs. (II) and (III) with

$$f \rightarrow V, \; B \rightarrow -L \quad \text{and} \quad A \rightarrow -G \quad .$$

The kernel T has no analog and is replaced by unity in the Gelfand-Levitan-Marchenko theory.

It is plausible that an analogy should exist between image improvement and the inverse scattering problem, since in both problems the aim of the theory is to determine the structure of a scattering medium by

observing its effects on the waves that pass through it. However, the preciseness with which this analogy is reflected in the structure of our equations came as a surprise. The logical basis of the analogy remains to be explored in detail.

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